The search for Goldbach's conjecture proof: Exploring your achievements

https://doi.org/10.56238/sevened2024.026-018

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ABSTRACT

Christian Goldbach was a Russian mathematician who lived in the eighteenth century. Although he saw mathematics more as a hobby, he published several works on Number Theory. It is in one of his correspondences with Euler that his conjecture arises, which states that every even number greater than 2 can be expressed as the sum of two prime numbers (equal or not). Despite having a very simple statement, its demonstration requires advanced mathematical knowledge, so much so that to this day it remains without a definitive proof. Obviously, several mathematicians have devoted many years of their lives to the study of Goldbach's conjecture. In this sense, the objective of this research is to list the advances achieved by such mathematicians over time. In addition, we will seek to better understand the historical context in which the conjecture arises. In the end, we will try to highlight the most significant results that at first can be the basis for further research.

Keywords: Goldbach Conjecture, Number Theory, Prime Numbers, Riemann Hypothesis.

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INTRODUCTION

Number Theory, a fundamental branch of mathematics, focuses on the study of integers and their properties. Its legacy dates back to Ancient Greece, around 500 BC, when Greek thinkers were already paying attention to this field of knowledge. Over the centuries, Number Theory has not only remained relevant, but has also established itself as a dynamic and challenging field of research, which continues to inspire and captivate current generations of mathematicians.

A distinctive feature of this area is the existence of problems that, despite their initially simple formulation, require evidence of significant complexity. Among these problems, we can mention the famous last Fermat's theorem, proposed by the French mathematician Pierre de Fermat (1607-1665) in the seventeenth century, which states that there are no integer solutions to the equation xn + yn = zn, in which $n > 2$ and $x, y, z \neq 0$. It was only proven at the end of the twentieth century, after more than 350 years of attempts. Other problems, however, continue to challenge the mathematical community, such as the Collatz conjecture, proposed in 1937 by the German mathematician Lothar Collatz (1910-1990): let $n > 0$ be a natural number. If n is even, divide it by 2, getting n/2. If n is odd, multiply n by 3 and add 1 to $3n + 1$. Repeat this process for the value obtained, and so on. The conjecture tells us that the final value obtained will always be equal to 1.

An extremely relevant concept in this area is that of the prime number: a positive integer $p \geq 2$ is called prime if its only positive divisors are 1 and p. Some of the most intriguing mathematical problems are, directly or indirectly, linked to these numbers. One of them will be the object of study of this research, which is the Goldbach conjecture, which appeared in 1742 and which today, after more than 280 years, remains without a definitive demonstration. According to this conjecture, every even number greater than 2 can be written as the sum of two prime numbers.

In this sense, the objective of this research is to list the results obtained so far in an attempt to prove the conjecture. In the end, we will highlight those that, at least at first, seem to be more interesting for future research. In addition, we will seek to understand the historical context in which the conjecture arises. In order to achieve our objectives, we focused on the original works of mathematicians who dedicated themselves to the study of conjecture, as well as other works (books, monographs, dissertations and theses) that dealt with the subject.

THEORETICAL FRAMEWORK

GOLDBACH'S CONJECTURE

Christian Goldbach (1690-1764)

Christian Goldbach was born on March 18, 1690, in Königsberg, Prussia, present-day Kaliningrad, Russia, and died on November 20, 1764, in Moscow. The son of a Protestant pastor, Goldbach studied mathematics, but mainly law and medicine.

In 1710, at the age of twenty, he began one of several trips through Europe, establishing contact with many of the leading scientists of the time. During this trip, in 1711, he met Leibniz (1646-1716), in the city of Leipzig. For the next two years, the two exchanged correspondence in Latin. In 1712, in London, Goldbach met the French mathematician De Moivre (1667-1754) and the Swiss Nicolaus (I) Bernoulli (1687-1759) who, like Goldbach, was also traveling in Europe.

Goldbach continued his long journey and was in Venice in 1721. Here he met Nicolaus (II) Bernoulli (1695-1726), who was also traveling in Europe. It was at the suggestion of Nicolaus that Goldbach began in 1723 a correspondence, which lasted seven years, with Daniel Bernoulli (1700-1782), the younger brother of Nicholas.

In 1724 Goldbach returned to his hometown, and met two mathematicians who influenced his life: the German Georg Bernhard Bilfinger (1693-1750) and the Swiss Jakob Hermann (1678-1733). Georg Bilfinger and Jakob Hermann were heading to St. Petersburg to help create the Imperial Academy of Sciences (later called the St. Petersburg Academy of Sciences), which would be organized (at Leibniz's suggestion) along the same lines as the Berlin Academy of Sciences.

In July 1725 Goldbach wrote to Laurentius Blumentrost (1692-1755), president of this new Academy, asking for a post. After an initial rejection, Goldbach was invited to the positions of professor of mathematics and historian in St. Petersburg. This invitation was due to the fact that Goldbach was already a well-known mathematician at this time, since 1717 he had been publishing works. It is worth noting that in 1717, after reading an article by Leibniz on the calculation of the area of a circle, he ended up focusing again on the theory of infinite series and in 1720 he published *Specimen methodi ad summas serierum in Acta eruditorum*. Goldbach was recording secretary at the opening ceremony of the Academy, held on 27 December 1725, and continued to hold this capacity until January 1728.

On May 17, 1727, the Swiss mathematician Leonhard Euler (1707-1783) arrived in St. Petersburg to take up a position at the Academy of Sciences. From this, he began a correspondence with Goldbach that would last about 35 years (between 1729 and 1764).

Of this correspondence, 196 letters have survived. Many of these letters dealt with various problems of number theory, some of them presented earlier by Pierre de Fermat. The extensive correspondence between Goldbach and Euler is a source of information on the history of mathematics in the eighteenth century, as it provides a fundamental record of Euler's legacy in number theory, even more so than Euler's own publications. Goldbach, even looking at Mathematics as a playful activity, developed an important work in Number Theory.

Historical context of the conjecture

On June 7, 1742, in one of these letters, Goldbach suggests the following, later exemplifying:

> I do not consider it useless to note also those propositions which are very probable, though there is no real demonstration, because even if they are afterwards found to be false, they may still furnish an opportunity for the discovery of a new truth. [...] So I want to propose a conjecture: Every integer that can be written as the sum of two primes can also be written as the sum of as many primes as you want (including 1), until all terms are unitary. (FUSS, 1843, p. 127).

In the margin of this same correspondence he further proposed a conjecture that would become known as Goldbach's "marginal" conjecture:

Every integer greater than 2 can be written as the sum of three primes.

At that time, Goldbach, as well as other mathematicians, considered the number 1 a prime — a convention that was later abandoned, as we well know. Thus, for numbers 3, 4, and 5, this first version of the marginal conjecture would be invalid. Thus, the accepted form of the conjecture — that is, not considering 1 a prime number — is:

Every integer greater than 5 can be written as the sum of three primes.

Or, analogously:

Every integer greater than or equal to 6 can be written as the sum of three primes.

Still following this letter, Goldbach comments that, from his marginal conjecture, he can obtain the following:

Every integer even greater than 2 can be written as the sum of two primes.

Euler, on June 30, 1742, effectively agrees that the marginal conjecture could be decomposed into two. The first, stated as follows:

Every even number greater than 2 can be written as the sum of two prime numbers.

(Goldbach's "strong", "even" or "binary" conjecture).

For smaller numbers, there is no need to resort to more complex methods/resources to find two prime numbers (equal or not).

However, as larger numbers are considered, the use of more elaborate mathematical methods, as well as the use of technological resources, become indispensable. In this sense, verifying the validity of the conjecture for the number 66.780.242 using only "pencil and paper", for example, is unfeasible, and it is therefore essential, in this case, to use technology.

It is not difficult to imagine, however, that if we take even higher numbers, even for technology the mission will become arduous. This is mainly due to two factors: the infinity of numbers and the unpredictability of prime numbers.

The second part of the marginal conjecture tells us that:

Every odd number greater than 5 can be written as the sum of three prime numbers.

This, more precisely, can be rewritten as:

Every odd number greater than 7 can be written as the sum of three odd prime numbers. (Goldbach's "weak", "odd" or "ternary" conjecture).

This conjecture (already proved) was thus recognized by the fact that, if the strong conjecture is proved, it is automatically also proved, since it becomes a corollary of the strong one. In fact, taking Goldbach's strong conjecture as true and since $N > 7$ is any odd number, then $N - 3$ is trivially an even number and can be expressed as $N - 3 = p1 + p2$ (according to the strong conjecture), where p1 and p2 are prime. Therefore, $N = p1 + p2 + 3$. In other words, the strong conjecture implies the weak one, but the weak one does not imply the weak.

Goldbach's (strong) conjecture, which is part of problem 8 of the list proposed by the German mathematician David Hilbert (1862-1943) during the Second International Congress of Mathematics on August 8, 1900, in Paris, has fascinated — and frustrated — many mathematicians and curious people for the past 280 years. In the last century, especially, as will be listed below, many works aimed at proving the conjecture have been developed, some of which have relatively interesting results. Even if such advances are still insufficient, this search has, as Sousa (2013, p. 39) points out, "contributed to the development of Number Theory itself, to the extent that other results have emerged, less important than the conjecture, but which may allow, perhaps, to prove it". The author also emphasizes that the investigation of the conjecture has "allowed the development of methods useful to Number Theory and other areas of Mathematics".

ABOUT THE RIEMANN HYPOTHESIS

Bernhard Riemann (1826–1866) was a German mathematician. In 1851, at the University of Göttingen, he obtained his doctorate, supervised by Gauss. His thesis was developed in the field of complex function theory.

In 1859, after Dirichlet's death, he was appointed full professor at Göttingen. Riemann provided a rigorous definition of the integrability of a function through the Riemann integral. He also generalized all geometries, including Euclidean, to Riemannian geometry, which served as the basis for Albert Einstein's theory of relativity.

His only work on number theory was published in 1859, entitled *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the number of prime numbers less than a given value). In it, from an identity previously discovered by Euler, he arrived at a function that became known as the Riemann Zeta function, which is a special complex variable function. In writing this work, Riemann's goal was essentially to set the path toward proving the Gauss conjecture about the distribution of prime numbers, which did not happen until a few years later. The famous Riemann hypothesis states that all non-trivial Zeta zeros are located on the critical line.

The Riemann hypothesis, associated with the Goldbach conjecture and the twin prime conjecture, make up the eighth problem in the list of 23 problems proposed by the German mathematician David Hilbert (1862-1943) during the International Mathematical Congress that took place in Paris in 1900. It is also one of the Problems of The Millennium Prize Problems of the Clay Mathematics Institute. The prize for those who present proof is 1 million dollars.

More than 12 trillion non-trivial zeros have been computationally found. In addition, in 1914, the British mathematician G. H. Hardy (1877-1947) showed that there are infinite zeros with a real part equal to 1/2. Despite this, the question remains open, more than 160 years later.

Many results in mathematics have been proven by assuming Riemann's hypothesis to be true, including some related to Goldbach's conjecture, as we will see later.

METHODOLOGY

The initial stage of the research that resulted in this work began with the choice of the theme. Once defined, the objectives were established and the methodology was selected, opting for bibliographic research.

According to Gil (2002, p. 44), the bibliographic research "[...] it is developed based on material already prepared, consisting mainly of books and scientific articles". Complementing this, Severino (2007, p. 122) tells us that bibliographic research is carried out by:

> [...] available record, resulting from previous research, in printed documents, such as books, articles, theses, etc. Data from theoretical categories already worked on by other researchers and duly recorded are used. The texts become sources of the themes to be researched. The researcher works from contributions from the authors of the analytical studies contained in the texts. (SEVERINO, 2007, p.122).

In this way, the texts chosen by the researcher play a fundamental role in providing the basis and support necessary for the elaboration of the work. They not only broaden the researcher's understanding of the topic, but also clarify the limits and possibilities of research, allowing for a more in-depth and comprehensive analysis.

In this sense,

> This research aims to know the different forms of scientific contribution already made on a given subject, aiming to enter current and relevant data on the investigated theme. It uses exclusively material already prepared and available, in particular books and scientific articles, and is the basis for any type of research [...] (METRING, 2010, p. 64).

After the bibliographic survey, a selection of the most relevant texts/materials to the research theme was carried out. With the systematic reading of this material concluded, the writing of the present research was carried out, integrating the readings related to the theme studied.

RESULTS AND DISCUSSION

In this chapter, we will detail the results of investigations done over time in an attempt to prove Goldbach's conjecture. The first attempts date back to the nineteenth century, but the most significant results were achieved only from the twentieth century onwards.

Between the date of the emergence/publication of the conjecture and the first result achieved, there lived, evidently, great mathematicians (considered by many, including some of the most prolific in history). Among them we can mention: the Germans Carl Friedrich Gauss (1777-1855), Ernst Kummer (1810-1893) and Richard Dedekind (1831-1916); the Italian Joseph Louis Lagrange (1736-1813); the Frenchmen Joseph Fourier (1768-1830), Pierre-Simon Laplace (1749-1827), Augustin-Louis Cauchy (1789-1857) and Henri Poincaré (1854-1912) and the Norwegians Niels Henrik Abel (1802-1829) and Sophus Lie (1842-1899). History does not tell us exactly whether such mathematicians ever knew of the existence of the conjecture, since, as Décaillot brings us:

> Considering that Goldbach's conjecture was not published until Waring (English mathematician) published it in 1770 (without attributing authorship, a proof, or commentary) in his work *Meditationes Algebraicae*, it is understandable that it is difficult to assess how much empirical work has been done on Goldbach's conjecture in the hundred years or so after its first appearance in Euler's letter. In fact, as mentioned above, until 1894, Poincaré was still asking where Goldbach's conjecture had been published. (DÉCAILLOT, 2008, apud FARRUGIA, 2018, p. 31).

What is known in fact is that Euler stated that he was practically sure that the conjecture was valid, but he would not, at least at that moment, be able to prove it.

SECULUM XIX

The nineteenth century was a period of great changes and transformations in world history. It was also in this century that the first studies related to the Goldbach conjecture began to appear.

In 1855, the French mathematician Louis-Charles-Antoine Desboves (1792–1883) published a paper in which he checked Goldbach's conjecture up to 10,000. His work was a significant advance in the search for a proof of this conjecture, as it is the first known study.

Later, in 1894, the great German mathematician Georg Cantor (1845–1918) provided a table with the decompositions of all even numbers up to 1,000 as the sum of two prime numbers.

Still at the end of the century, the German mathematician Carl Hermann Robert Haussner (1863-1948) shows the validity of the conjecture first up to 100,000 and then up to 1,000,000.

Although the results achieved in the nineteenth century were mathematically of little importance, they were relevant because they inspired other mathematicians to research the subject.

TWENTIETH CENTURY

In the twentieth century, great mathematicians also lived, such as the Germans David Hilbert and Emmy Noether (1882-1935); the Austrian Kurt Gödel (1906-1978); the British Alan Turing (1912-1954) and Bertrand Russell (1872-1970); the Hungarian John von Neumann (1903-1957); Frenchman André Weil (1906-1998); the Chinese Shiing-Shen Chern (1911-2004); Americans Paul Cohen (1934-2007), Serge Lang (1927-2005) and John Forbes Nash (1928-2015); the Indian Srinivasa Ramanujan (1887-1920). Great Brazilian mathematicians also emerged and worked actively in this century, such as Lélio Gama (1892-1983), Maurício Peixoto (1921-2012), Manfredo do Carmo (1928-2018), Jacob Palis (1946-), Marcelo Viana (1958-) and the renowned Elon Lages Lima (1929-2017).

It was from this century that important studies began to be carried out and disseminated, with the aim of proving Goldbach's conjecture (or, at least, increasing the arsenal of tools to begin to "attack" the problem, which until then was practically non-existent). It is important to emphasize that, as Wang (2002) explains, the great achievements of Analytic Number Theory in the nineteenth century, in particular, the theory of Chebyshev (1821-1894), Dirichlet (1805-1859), Riemann, Hadamard (1865-1963), de la Vallée Poussin (1866-1962) and von Mangoldt (1854-1925) on the distribution of prime numbers, were essential for the advancement in the study of conjecture.

The statements of these results are, as the reader may already imagine, too complex; the vast majority with dozens of pages. Because of this, we will not explain in detail the elements present in each of them, because doing so would be beyond the objective of this work. We leave, however, as a reading suggestion, the book The Goldbach Conjecture, by Yuan Wang, in which these results are detailed.

The first major result appears between 1919 and 1920, when the Norwegian mathematician Viggo Brun (1885-1978), in his work entitled *Le crible d'Eratosthène et le théorème de Goldbach*, demonstrated that *any sufficiently large even number is the sum of two numbers, each having a maximum of nine prime factors.*

To reach this conclusion, what Brun basically does is modify the Sieve of Eratosthenes, based on the work of the young French mathematician Jean Merlin (1891-1915), killed in combat during the First World War and whose work was published posthumously by the French mathematician Jacques Hadamard. Through this sieve, information is obtained about pairs of numbers whose sum is an even number.

In 1923, Hardy and Littlewood (1885-1977) showed in their work *Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes*, assuming the Riemann hypothesis, that *every sufficiently large odd number is the sum of three prime numbers and that* *almost all even numbers are sums of two primes*. Littlewood, at the time, estimated that this "large enough" number should be greater than or equal to 1050.

Hardy and Littlewood's demonstration, although of great ingenuity, had two aspects that limited it: one was the fact that it depended on the Riemann hypothesis; the other is that it did not determine how large the number in question should be. As we will see later, these limitations have been reduced.

Nevertheless, Hardy and Littlewood were the pioneers in building the knowledge we have about the Goldbach conjecture.

Three years later, the German mathematician Bruno Lucke (1902–1944), under the supervision of Edmund Landau (1877–1938), determined in his doctoral thesis that the number 1032 could already be considered "sufficiently large" for the Hardy–Littlewood theory.

Based on Brun's result, the Russian Lev Schnirelmann (1905-1938) proved in 1930 that *every even number greater than or equal to 2 can be written as the sum of no more than 20 prime numbers*. This result would be perfected by the end of the century.

In 1932, the British mathematician Theodor Estermann (1902-1991) improved Brun's result by proving that *every sufficiently large even number is the sum between a prime number and a number that is the product of a maximum of 6 prime factors*. Estermann proved this, however, using the Riemann hypothesis.

Five years later, the Russian Ivan Vinogradov (1891-1983), in his work *Representation of an odd integer as a sum of three primes* , achieved a great result: he managed to remove the dependence on the Riemann hypothesis, thus generating the unconditional proof of Hardy's and Littlewood's results — but he could not determine what "big enough" meant. Later, Russian mathematicians Yuri Linnik (1915-1972) and Nikolai Georgievich Chudakov (1908-1986); the American Hugh Lowell Montgomery (1944-); the Chinese Pan Cheng Biao (1938-); Britain's Mary Frances Huxley (1924–2014) and India's Kanakanahalli Ramachandra (1933–2011) presented alternative proofs for this theorem.

Vinogradov's result is a remarkable advance, as it removed one of the main limitations of Hardy and Littlewood's proof, which was dependence on the Riemann hypothesis. Once this was done, most of the following studies aimed to define from which value a number would be considered "large enough".

In 1938, Chudakov proved, as did Littlewood, that almost all even numbers are sums of two primes, showing that the distribution of prime numbers is very regular, even on large scales.

The Hungarian mathematician Alfréd Rényi (1921-1970) provoked, 16 years later, the result of Estermann disregarding the Riemann hypothesis.

In 1956, however, the mathematician K. G. Borodzkin, a student of Vinogradov, showed that the fact that he removed the Riemann hypothesis would have a problem: "big enough" in his proof would be numbers greater than 107,000,000. This value was later improved, but it is still absurdly higher than the values found by Lucke and Littlewood. Notice, then, that using the Riemann hypothesis in this context considerably decreases the value from which a number will be considered "large enough".

The 1960s, marked by several events around the world, such as the beginning of the Vietnam War, the assassinations of Martin Luther King Jr. (1929-1968) and Robert F. Kennedy (1917-1963) and the arrival of man on the moon, was also remarkable in terms of advancing knowledge about the Goldbach conjecture.

Brun's result (already optimized before) was further improved in 1962, when the Chinese mathematician Wang Yuan (1930-2021) (assuming the Riemann hypothesis) states that *every sufficiently large even number is the sum between a prime number and the product of at most 3 prime factors*. In the same year, this result was also verified for the case in which one of the numbers is the product of a maximum of 5 factors, by the also Chinese Chengdong Pan (1934-1997), and for the case in which one of the numbers is the product of a maximum of 4 factors, by the Russian Mark Borisovitch Barban (1935-1968).

But it is in 1966, in a work entitled *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, that the result considered the most important achieved to date appears: the Chinese mathematician Chen Jingrun (1933-1996), surpassing Wang, Pan and Barban, who 4 years earlier had achieved incredible results, He demonstrated that *every sufficiently large even number is the sum of a prime number and a product of at most two primes.*

Chen's result is considered the most successful so far because of two aspects: the first is the fact that he does not use the Riemann hypothesis; the second, because it is the closest statement to Goldbach's strong conjecture.

To reach this conclusion, Chen used, in addition to the existing work on conjecture, the following theorems, which are specific to Number Theory: the Prime Number Theorem, the Bombieri–Vinogradov Theorem, the Siegel–Walfisz Theorem, and the Jurkat–Richert Theorem.

In 2006, a bronze statue was erected in honor of Chen on the campus of Xiamen University in Fuzhou, China. It is a representation of their hard work and dedication to their research. In addition, a commemorative stamp was launched, a collector's item for the country's mathematicians.

Almost 10 years after Chen's result, one emerges that could have been a "watershed" in the search for the proof of Goldbach's conjecture, which is the theorem of Montgomery and Robert Vaughan (1945-): *let E(X) be the function that counts even numbers not exceeding X, which cannot be written as the sum of two primes. So,* $E(X) \ll X^{1-\delta}$*, where* δ *is a positive constant*. The problem with this statement, as it is easy to imagine, lay in the fact that X and δ were unknown. Despite this, Montgomery and Vaughan's result tells us that the probability of a number not being representable as the sum of two primes is very small. In 1980, 5 years later, the works of Chengdong Pan and Chen estimated that $\delta > 0.04$.

In 1989, mathematicians Chen (again) and T. Z. Wang (1933-2017) reduced the number found by Borodzkin in 1956 to 1043,000. Seven years later, they decrease it again to 107,194. However, checking the other cases, even for the most modern computers, is impossible.

Schnirelmann's result of 1930 was improved in 1995 by the French mathematician Olivier Ramaré (1965 -), who was able to prove that *every even number greater than 2 can be written as the sum of a maximum of 6 prime numbers.*

At the end of the 90s, more precisely in 1997, we have the penultimate result obtained in the twentieth century: the French mathematicians Jean-Marc Deshouillers (1946-) and Gérard Effinger (1945-2019) and the Dutchman Hendrik te Riele (1947-) showed that *the Riemann hypothesis implies Goldbach's weak conjecture, for all odd* numbers. In other words, they demonstrated Goldbach's weak conjecture from results that were previously proven by assuming the Riemann hypothesis.

In 2000, the last of the century appears: the Asian mathematician Hongze Li determines that in the theory of Montgomery and Vaughan, $\delta \approx 0.08622$.

TWENTY-FIRST CENTURY

Contrary to what many think, there are currently important mathematicians who work incisively so that the existing areas of Mathematics continue to be explored, as well as new areas can be discovered, thus enabling Mathematics to be in constant evolution. In this sense, we can mention the Australian mathematician Terence Tao (1975-), the Russian Grigori Perelman (1966-), the British Ben Green (1977-), the Iranian Maryam Mirzakhani (1977-2017) and the Frenchman Cédric Villani (1973-).

Specifically with regard to Goldbach's conjecture, the first result appears at the beginning of the century, between 2001 and 2002, when the Chinese Ming-Chit Liu (1937-2023) and T. Z. Wang reduced the value that was until 1996 from 107,194 to 101,346.

A few years later, in 2010, Chinese mathematician Wen Chao Lu determines a more precise value for the constant δ of Montgomery and Vaughan's theorem: $\delta \approx 0.121$.

In 2012, Terence Tao proves that *every odd number can be written as the sum of a maximum of 5 prime numbers*.

Shortly thereafter, the Peruvian mathematician Harald Helfgott (1977-), until then not well known in the mathematical community, presented, after 7 years of research, an unconditional proof for Goldbach's weak conjecture, that is, a proof independent of the Riemann hypothesis.

The weak conjecture had been computationally verified in 2011 by the British David Platt for all odd numbers n, such that 7 < n ≤ 1, 23163 · 1027. Platt, however, did this by assuming the Riemann hypothesis. Later, in 2013, they verify, disregarding this hypothesis, that Goldbach's weak conjecture applies to all odd up to 8,875 · 1030. With this, what he does later is to show, analytically, that it also applies to the other odd numbers.

According to Helfgott (2013, p. 3), analytic proofs often establish a result for integers above a particular constant. In addition to demonstrating the existence of a specific numerical representation (such as the sum of three prime numbers), an analytic proof also provides an estimate of the weighted number of ways to realize that representation.

This achievement earned him the Humboldt Research Prize, awarded by the Alexander von Humboldt Foundation, in Germany, in addition to a sum of money.

In 2014, Polish mathematician Adrian Dudek proved that *every integer greater than two is the sum of a prime number and a square-free number*, which is a consequence of Estermann's 1932 result.

Chen's result, considered the most successful to date, was improved in 2015 by Japanese mathematician Tomohiro Yamada, who determined that "sufficiently large" in Cheng's proof would be any number greater than 1.7 · 101.872.344.071.119.343.

Undoubtedly, a mark of the twenty-first century is technology, which as he explains Santos, Medeiros and Ribeiro (2017, p. 83):

> In contemporary society, man shares space with machines, and the relationships constituted by both become increasingly intrinsic. Whether at home, at work, at school, in parks, in various spaces and social environments, technology has its place of usefulness, and is often a necessity. It is almost impossible to imagine the survival of man without technology today. The frenetic way in which children, young people and adults consume technologies and media in the current model of society is proof that the digital age has revolutionized the behavior, feelings, education, way of living, being and thinking of individuals.

This usefulness cited by the authors has also been seen directly in Mathematics. For example, calculations that could once take hours to complete are now performed in a matter of seconds. In addition, technology can also be taken to the classroom, a practice that has been increasingly common and necessary.

It has also been used to try to better understand the Goldbach conjecture, since it can also be studied by more mechanical methods, through the use of computers. In this method, one forcibly tries to find the two prime numbers that add up to each even number. Obviously, there is no computer capable of verifying this for all numbers, since these are infinite. The goal in employing this method is to try to find out if there is any rule (any pattern) in the behavior of these numbers. Or even find a counterexample, i.e., an even number that cannot be written as the sum of two primes, which would refute Goldbach's conjecture.

The best result achieved so far through technology is from Professor Tomás Oliveira e Silva, from the Department of Electronics, Telecommunications and Informatics at the University of Aveiro, Portugal. He computationally verified that *all even numbers up to 4·1018 can be written as the sum of two primes*. This, of course, is not in itself sufficient to prove the conjecture. However, it can be an important source for future computational research.

At the beginning of this decade, in 2020, mathematicians Forrest J. Francis and Ethan S. Lee establish conditions of divisibility on the free number of squares on Dudek's work. In 2022, the number found by Yamada is slightly improved. Mathematicians Matteo Bordignon, Daniel R. Johnston, and Valeriia Starichkova determined that "large enough" in Chen's result would be every number greater than . Shortly thereafter, assuming Riemann's hypothesis, they reduced this number to .$e^{e^{32,6}}$ $e^{e^{15,85}}$

After that, studies based on the results listed so far appeared, but which did not surpass them. For example, Daniel R. Johnston and Valeriia V. Starichkova (again), inspired by the results of Estermann and Rényi, showed that every integer even greater than 2 can be written as the sum of a prime and a number with a maximum of 369 prime factors. They also showed, under the assumption of the Riemann hypothesis, that this outcome can be improved for 33 prime factors. As already said, such demonstrations were not an advance in relation to what already existed.

Many supposed proofs of the Goldbach conjecture are circulated on the internet from all over the world. On the one hand, this demonstrates the interest of many amateur mathematicians in the conjecture. On the other hand, it can also lead to the dissemination of false or incomplete information. Obviously, all were disregarded for being flawed or based on false premises.

The fact is that the importance of this problem for the mathematical community goes beyond the theoretical field. If it is ever demonstrated, the mathematician who solves it will certainly be recognized and rewarded by his peers, in addition to experiencing indescribable personal satisfaction.

FINAL CONSIDERATIONS

It is undeniable that there has been significant progress in the search for a proof of the Goldbach conjecture. Naturally, some of these advances stand out for their relevance, among which we emphasize the results obtained by Chen and by Montgomery and Vaughan.

The first, accompanied by its later refinement recently achieved, is the closest mankind has come to a proof for the Goldbach conjecture. This combination then results in the following theorem:

Chen's theorem: *Any even number that can be written as the sum of a prime number is the product of a maximum of two prime numbers.*$e^{e^{32,6}}$

The importance of this theorem is not restricted to Goldbach's conjecture, since the path taken until him allowed the development and improvement of other mathematical theories.

Although it is a great result, it is indisputable that there is still a long way to go. In fact, notice that even if Chen's Theorem (refined) had been further refined and told us today that *any even number greater than that can be written as the sum of two primes*$e^{32,6}$, this would still not be enough for us to say that the mathematical community would be close to the longed-for proof for the Goldbach conjecture.

Remember that there would still be a giant amount of numbers between 4 · 1018 is to be verified, with many possessing an unimaginable amount of digits. It is a task that requires hard work and, unfortunately, we cannot say anything precise about the current level of interest/commitment on the part of mathematicians, especially.$e^{e^{32,6}}$

Another result that deserves to be mentioned here is that of Montgomery and Vaughan, from 1975. They proved by recalling that *since E(X) is a function that counts even numbers not exceeding X, which cannot be written as the sum of two primes, then* $E(X) \ll X^{1-}\delta$, where δ is a positive constant. Today, the best approximation we have for the constant is $\delta \approx 0.121$. In this way, the above statement can be rewritten as:

Vaughan–Montgomery theorem: *Let E(X) be the function that counts even numbers not exceeding X, which cannot be written as the sum of two primes. So* $E(X) \ll X\ 0.879$. This theorem can also be an interesting option to refute the conjecture. Here, the concern is not to determine exactly the number or numbers that cannot be expressed as the sum of two prime numbers, but rather to verify the existence of these numbers. For this, of course, it would be necessary to have a better knowledge about the value of X; we know, although at first it does not help so much, that X > 4 · 1018, according to the work of Oliveira e Silva. Also, notice that if there is another approximation for δ, specifically with δ > 1, the power $X^{1-\delta}$ would tend to 0 as an increasing X is considered. This would prove Goldbach's conjecture, since $E(X) \ll 1$.

More recent work, such as Dudek's, which proved that *every integer greater than two is the sum of a prime number and a square-free number*, require substantial improvement. In Dudek's research, for example, the list of square-free numbers is infinite, which means that his result still needs improvement to be really useful in the search for the proof of the Goldbach conjecture.

Certainly, it is essential to emphasize the significance of the confirmation of the weak conjecture in Mathematics, particularly in Number Theory. Although only a minority of his results can be applied to a potential proof of the strong conjecture, Helfgott's validation of the simpler version played a crucial role in re-attracting some interest from the mathematical community towards the more challenging conjecture.

We strongly believe, therefore, that this research achieved its objectives. First, because we were able to list the results existing in the tireless search for proof of this conjecture; second, because we understand historically how it came about, and finally, because we highlight those results that seem to be most promising in future research.

We also recognize that this research, like any other, has its limitations. The biggest one is perhaps the fact that we have not mathematically detailed the process of constructing at least some of the results, which can be explained in future research. On the other hand, from the academic point of view, it contributed to the diffusion of this problem, which is still little discussed in our literature, especially with regard to the list of existing attempts.

Despite the intense efforts dedicated so far by mathematicians, we conclude this work with the belief that a few decades of research, at least, will still be necessary to reach a definitive proof of Goldbach's conjecture, a feeling that is corroborated by Helfgott himself in a contact made via e-mail, when he stated that: "I feel that Goldbach is still far. There has been significant progress this century towards twin primes and also towards Chowla's conjecture, but it may be that those problems, long considered to be of equal difficulty, indeed 'morally equivalent', are in fact somewhat easier. Or perhaps progress towards them will stall". ("I feel that Goldbach is still far away. There has been significant progress in this century in relation to twin primes and also in relation to the Cohla conjecture, but it may be that these problems, long considered to be of equal difficulty, in fact 'morally equivalent', are a little easier. Or perhaps progress toward them stagnates." - Our translation).

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