# Multi-objective optimization using evolutionary algorithms: An antifragile approach to portfolio selection

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## ABSTRACT
Considering the documented difficulties in empirically applying widely recognized portfolio selection models that use mean-variance relationships as central features, as proposed by Markowitz and later extended in CAPM by Sharpe, this study aims to extend these theories. Drawing upon contributions from Keating and Shadwick, which highlight CAPM's limitations in handling non-normal distributions, the study introduces non-convex attributes into a multi-objective optimization framework using evolutionary algorithms. Additionally, an antifragile metric known as CVIX is implemented to assess conditional correlation with the VIX, thereby addressing questions concerning the feasibility of market portfolios outperforming CAPM's theoretical market portfolio. Optimizations were carried out on U.S. markets, using time frames from 1994 to 2022. The results are encouraging; in contrast to optimizations that employed solely convex attributes, which yielded inferior outcomes in all scenarios compared to the OCAPM model, applying the antifragile metric along with non-convex attributes in multi-objective optimization produced superior results.

**Keywords:** Multiobjetive Optimization, Evolutionary Algorithms, Antifragility, CAPM, OCAPM.

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## INTRODUCTION

With regard to the optimization of portfolios and the formation of efficient portfolios, that is, those that seek to maximize returns and mitigate the risks associated with assuming the investor's rationality, it is essential to talk about the theoretical framework left by Markowitz (1952). He inaugurated the modern theory of finance by understanding the logic behind investors' pursuits, associating these desires with two central attributes: the mean and the variance. Markowitz conceptualized his diversification method based on the covariance between assets, giving rise to the theory of portfolios and the efficient frontier.

Based on this, studies such as that of Sharpe (1964) sought to expand this concept in optimization. Sharpe (1964) then proposed the CAPM when he identified that the diversification proposed by Markowitz (1952) dealt well with the reduction of non-systematic risk, but was not capable of quantifying systematic risk. Therefore, it was not possible to measure the overall risk-adjusted performance of the portfolio.

In order to construct a metric capable of quantifying these attributes that were not priced in Markowitz's (1952) original model, Sharpe (1964), in the CAPM model, which also had important contributions from other authors such as John Lintner (1965), introduced optimization as a risk-free asset. According to the author, under perfect conditions, the expected returns on assets would be represented by the return of a risk-free asset, with an increase relative to the risk assumed. To consider this relationship, he proposed the use of the Beta coefficient, which will be used to measure the risk sensitivity of the asset, thus being able to quantify the systematic risk and the contribution of the asset to the overall risk of the portfolio.

However, despite being widely recognized and accepted in the world of finance for offering a robust theoretical foundation, its practical application has undergone validation tests over all these years. The model presents difficulties in its empirical application, first due to the need to work with past results for its weightings. It is necessary to use wallet proxies that will only be able to estimate if the portfolio is on the minimum variance border. Second, as Galagedera (2007) points out, many studies note that CAPM in its basic form may not fully explain the variation in expected returns on assets. For this reason, it has given rise to a continuous flow of research looking for alternative pricing models. As Assaf Neto (2012) points out, the current focus is more on improving CAPM than on replacing it.

Following this reasoning, Vasconcelos et al (2013), using the previous contributions of Keating and Shadwick (2002) and Kazemi et al (2004) that gave rise to the OPM (Omega Performance Measure), implemented this new measure to the optimization model of Sharpe (1964). The goal was to use Omega to change the composition of the Betas in order to go through all the moments of the distribution and fill gaps in the model.

Having contextualized the current paradigm in portfolio optimization based on the CAPM model, we arrive at the main objective of this article. The objective is, by benefiting from the theoretical framework left by all the authors previously mentioned in search of the theoretical and practical improvement of the CAPM, to evaluate the performance of single-objective optimizations with convex attributes. As well as the performance of multiobjective optimizations with the mediation made by evolutionary algorithms, at first only with the addition of the Omega measure and maximum *drawdown*, later, along with them another attribute: the CVIX. The ultimate goal is, therefore, to evaluate the feasibility of increasing the optimization of portfolios of these attributes by comparing their contributions.

## THEORETICAL REVIEW

The theoretical review was done by going through all the main theories and concepts used for the execution of the experiments and simulations of portfolios and, therefore, for the creation of this article as a whole. The theories and concepts used were: Portfolio Theory and Efficient Frontier both introduced by Markowitz (1952), the contribution of Sharpe (1964) which can be seen as a theoretical evolution of Markowitz since it uses the same bases with the introduction of the concept of systematic risk represented by the beta coefficient and the Omega measure in asset selection proposed by Keating and Shadwick (2002). We also listed some recent works using evolutionary algorithms for the selection of assets that we believe present relevant contributions to the theme.

### PORTFOLIO THEORY

The winner of the Nobel Prize in Economics in 1990, Harry M. Markowitz presented, long before his award, in 1952, the article *Portfolio Selection*, which introduced the world to a revolutionary perspective for the time on the formation of investment portfolios. Within the paper, both the theory of portfolios and the concept of the efficient frontier were developed and presented to the world. The central idea proposed by Markowitz says that investors should consider both the expected return and its variance in asset selection, the expected return is the metric that should be maximized, while the variance should be minimized through the diversification of assets with the lowest ratio of covariance between them.

Markowitz (1952) emphatically states that the simple search for maximizing the expected return is a mistake, since applying this criterion opens the possibility of selecting 2 or more assets with similar returns, which, without the evaluation of their covariances, can represent a significant increase in the overall risk of the portfolio. The author also adds that the choice of a set of assets that present the maximum expected return will not necessarily have the lowest associated risk, because, as previously mentioned, mere diversification without prior evaluation of the covariance between the selected assets, considering only the number of assets included in the diversification, will not necessarily reduce the risk associated with the portfolio.

Thus, the greatest contribution of Markowitz's (1952) model to the theory of portfolios was definitely the diffusion of the understanding that diversification through assets with reduced levels of covariance is indispensable for the construction of a portfolio capable of mitigating variance, that is, the reduction of the risk associated with expected returns.

As stated earlier, the two main points of concern of portfolio theory are expected return and variance, and within the Markowitz model the mathematical representation of these concepts is done as follows:

The expected return Rp of a portfolio of assets ai, i=1, 2, ...., n, is expressed by its mean, given by:

$$R_P = \sum_{i=1}^{n} w_i \cdot R_i$$

where Wi is the weight or share of each asset in that portfolio and Ri is the expected return for each of the assets.

The second metric considered, the risk represented by the variance or similarly through its standard deviation σρ and mitigated through the calculation of the correlation coefficient or covariance of the assets and can be expressed mathematically as follows:$\sigma\rho^2$

$$\sigma_P = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}}$$

where σij represents the covariance between the active αi and αj, which means that σii is the variance of the active αi itself.

Thus, through technological advances and the emergence of quadratic programming, it became possible to build the portfolio suggested by Markowitz, because once the expected returns and expected variances for each asset have been calculated, as well as the expected covariance for each pair of assets and thus varying the Wi compositions, it is possible to build all possible portfolios with the set of selected assets that must respect the non-leverage constraint. The non-leverage constraint implies that the sum of the asset weights is less than or equal to 1 and greater than or equal to 0, this constraint can be viewed mathematically through the following formula:

$$\sum_{i=1}^{n} w_i = 1 \text{(todo o capital deve ser aplicado)}$$

$$0 \leq w_i \leq 1 \text{ (não alavancagem)}$$

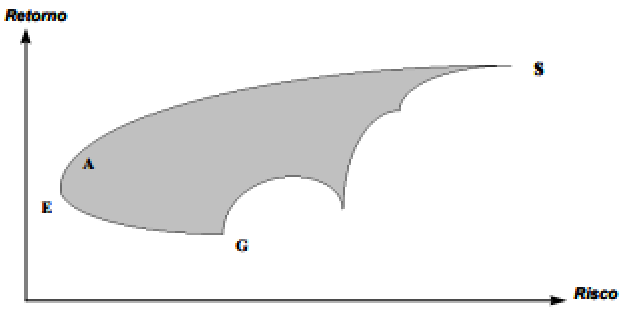
The contributions made by Markowitz (1952) continued to be relevant after all this time, and changed the way the formation of portfolios was seen forever, and served as the basis for other important advances in the area, as stated by Tambosi Filho e Silva (2000, p.1) "After the work of Markowitz (1952), entitled *Portfolio Selection*, several other works have emerged, with the proposal or simplification of the original formulation". Sharpe's CAPM (1964) is an example that extended the model, and continues to be used as a theoretical foundation even by this article, where the premises were applied in order to overcome the difficulties of its practical application and the lag it suffered over time. Portfolio theory also gave rise to the concept of efficient frontier that will be worked on in the next topic, and is complementary to this section.

## EFFICIENT FRONTIER

To understand the formation of the efficient Markowitz frontier, it is necessary to retrieve some concepts presented in the theory of portfolios, since the frontier is the product of the method of combinations of assets and portfolio formations previously presented. Following this reasoning, it is of paramount importance to understand that the efficient frontier is the result of the implementation of the variables that were prioritized by Markowitz (1952), the expected returns represented by the calculation of their averages taking into account the weights associated with each asset, the risk expressed by the variance of these returns, and not least the expected covariance for each pair of assets. From this, it is possible to estimate the expected returns and the variances and covariances for all possible combinations of portfolios with the selected group of assets by varying the weights associated with each asset, giving rise to a set of hyperbolas.

In addition, the formation of Markowitz's (1952) portfolio of risk assets must also respect the non-leverage restriction, which consists of the sum of the weights being equal to or less than 1 and equal to or greater than 0, in this way it is possible to transform the previously mentioned set of hyperbolas into a compact set that will give rise to the so-called Feasible Region.

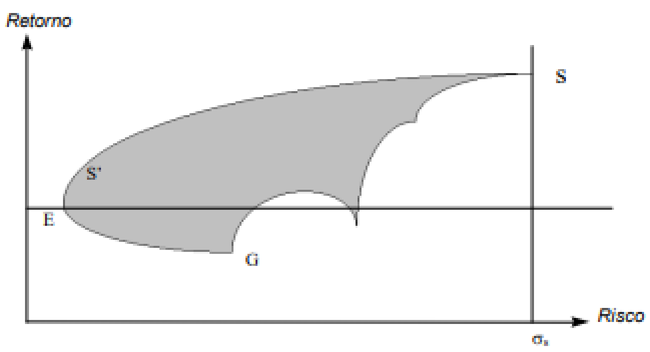
The obtainment of the feasible region can be graphically shown below:




Where the feasible region is represented specifically by the letter A which means that any point within that region is in accordance with the constraints, thus delimiting the area that interests us in the formation of portfolios.

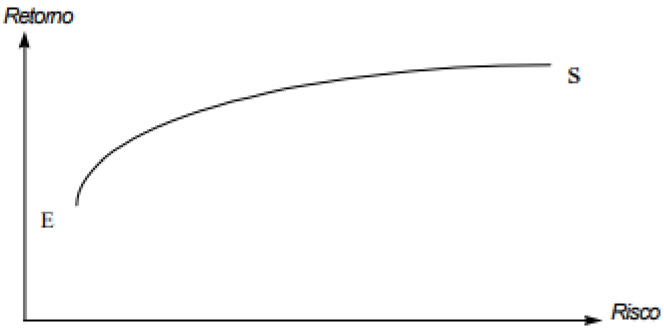
Now it is necessary to understand one more concept that is used in the formation of the efficient frontier, the principle of dominance, which is a multidisciplinary concept, but when applied to portfolio management can be understood as one portfolio that dominates the other if it offers a higher expected return for the same level of risk or if it offers the same expected return for a lower level of risk. The concept of the principle of dominance, as it has been described, was worked on both by Markowitz (1952) and by several other authors in the most different fields of knowledge, such as Howard Raiffa in *Decision Analysis* (1968).

Going back to Figure 1, if we apply the principle of dominance we will get the graph in Figure 2 and assuming that the investor is rational we will have 2 points of interest in the graph, point E which represents the minimum risk and point S which represents the point of maximum return.




Taking into account the primary objective of optimization (obtaining portfolios that offer the maximum expected return for different levels of risk) we can assume that for each portfolio with risk between σE and σS, the maximum return is situated at the upper border of the region S. It is ultimately Markowitz's (1952) efficient frontier.

The formation of the ES curve and consequently the efficient frontier can be seen below:




In this way, each point of the curve, that is, of the border, represents an efficient portfolio since it presents the highest return among those possible for that level of risk, which allows the investor to delimit the best possible combinations according to the metrics considered by Markowitz (1952), thus eliminating an infinity of portfolios that are not efficient in the risk-return ratio, facilitating decision making.

Thus, the investor will be able to choose the one that best fits his profile. It is important to emphasize that the formation of the efficient frontier does not depend on the investor's profile, being purely a logical mathematical relationship, the choice of the portfolio present within the frontier that does depend on the investor's risk profile as well as their objectives.

The mapping of these needs to be related to the efficient frontier can be done, for example, by means of the utility curve, where by superimposing the utility curves on the frontier in order to identify the one that will touch the ES curve, showing the point that represents the most appropriate portfolio for the investor in question.

## SHARPE'S CONTRIBUTION

William Sharpe, in 1964, proposed the CAPM (*Capital Asset Pricing Model*) based on the theory of Markowitz (1952), which was unraveled in the 2 previous topics. Thus, the CAPM model can be considered an extension of the model proposed by Markowitz, since Sharpe (1964) used the theoretical foundation proposed for the construction of his model, adding to the calculation the beta index and a risk-free asset (commonly represented by government bonds) in order to construct a metric capable of quantifying the risk-adjusted performance of a portfolio. He also had important contributions over the years from Jack Treynor (1962), John Lintner (1965), Jan Mossin (1966), Tobin (1958) and Black (1972).

The core of Markowitz's theory lies in the idea that investors have a microeconomic preference for return over risk. Sharpe sought to expand this understanding, realizing that although the diversification proposed by Markowitz (1952) dealt well with the reduction of non-systematic (microeconomic) risk, it was not capable of pricing systematic (macroeconomic) risk.

Following this reasoning, Sharpe, relying on the premises of both Markowitz (1952) and the utility theory and the market efficiency hypothesis, postulated that, under perfect conditions, the expected return of a risky asset is given by the expected return of a risk-free asset with an increase in the premium relative to the risk assumed. The premium would then be the result of the difference between the return of the risk-free asset and the market asset with associated risk. To consider this relationship, Sharpe (1964) proposed the use of the beta coefficient, which will serve to measure the sensitivity to risk that the asset has, therefore, measuring its associated systematic risk, which cannot be extinguished through the diversification previously proposed by Markowitz (1952). Thus, the CAPM model was created, which serves to measure whether the risk assumed is in accordance with the expected return, taking into account the asset's risk sensitivity, i.e., the associated unsystematic risk, thus making it possible to theoretically determine whether the asset's risk offers profitability compatible with the risk to which it exposes the investor.

The contribution of Sharpe (1964) thus gave rise to the conventional formula of the CAPM model:

$$R_I = R_f + \beta (R_M - R_f)$$

Where

- Ri represents the expected return of the asset i
- Rf represents the risk-free asset (such as the U.S. Treasury bond)
- B is the beta index (Indicator of sensitivity of the asset's return to the market's return, if B = 1 the asset tends to move in tune with the market, if B > 1 the asset tends to be more volatile than the market, and if B < 1, it has a lower sensitivity to market movements)
- Rm represents the expected return of the market

The ratio (Rm – Rf) can also be referred to as the risk premium, as it represents the expected reward for investing in a risky asset rather than the risk-free asset. Another factor to be observed is the linearity of the equation, which makes it possible to keep the variables Rf and Rm constant and using the same period of time to reach the conclusion that assets that have the highest beta index will present both higher returns and more significant losses, and the inverse relationship regarding the beta is also true. That is, the asset that has a lower beta will have lower returns as well as lower risks.

Despite being widely recognized and accepted in the world of finance, CAPM is not without its critics. Since its inception, the validity of the model has been repeatedly tested. As Assaf Neto (2012) points out, the current focus is more on improving CAPM than on replacing it. Many studies have noted that CAPM, in its basic form, may not fully explain the variation in expected asset returns. The existence of gaps in the theory has given rise to a continuous flow of research seeking alternative pricing models, as pointed out by Galagedera (2007).

## MEDIDA WITH ÔME

To deal with the omega measure, it is important to contextualize its emergence that will complement the previous topics and give a sense of continuity, because, as previously discussed when talking about the contribution of Sharpe (1964) and Markowitz (1952), despite providing a consistent theoretical framework to inaugurate the modern theory of finance, they still had gaps that over the years were the subject of study by several authors, such as Keating and Shadwick (2002), who observed that the model proposed by Markowitz (1952) presented complications when working with non-normal distributions, suggested the use of the Omega measure, which takes into account the entire format of the distribution of returns of the asset to assess its associated risk.

Based on this perspective, critics of the simplification that mean and variance would be able to fully describe the distribution of returns, began to use the Omega measure in order to be able to compute the total impact of the distribution, unlike Markowitz (1952) who estimated only two individual moments. Thus, the Omega performance measure formulated by Keating and Shadwick (2002) was defined by the definition of a limit external to the model, defined by the investor and which is usually the risk-free rate, this limit is responsible for dividing the distribution of return probabilities into two areas, the area of returns located to the right of the L (limit) and the area of losses to the left When defining the limit for the use of the measure, Omega Keating and Shadwick also postulated that it should be the minimum amount of gain expected by the investor, and for this reason the risk-free rate is constantly used because it is usually the comparative return index because it is a "guaranteed" return.

This reasoning gave rise to the original formula for the Omega performance measure, which is defined as:

$$\Omega(L) = \frac{I_2}{I_1} = \frac{\int_L^b [1 - F(x)]dx}{\int_a^L F(x)dx}$$

Where:

F = cumulative earnings distribution function

L = minimum required level of earnings

A = Minimum Return

B = Maximum Return

( a,b ) = Lower and upper limits respectively of the range of returns of the distribution. Most of the time a = -∞ and b = ∞

l2 (L) = The weighted average of earnings above L

l1 (L) = The weighted average of losses below L

In this way, using the Omega (L) function, it is possible to compare the expected returns of different assets and classify them in relation to their Omegas. Therefore, a higher Omega will indicate that the asset is a better investment, since Omega(L) = 1 means that the weighted gains equal the weighted losses, and one should always look for an Omega greater than 1 because this will indicate that the weighted probability of returns above the limit is greater than the weighted probability of returns below the limit.

The Omega measurement as previously presented can be demonstrably represented in an alternative way, this simpler form was postulated by Kazemi et al (2004) and the way in which Kazemi came to this conclusion can be visualized mathematically as follows:

$$\Omega(L) = \frac{\int_L^b [1 - F(x)]dx}{\int_a^L F(x)dx} = \frac{\int_L^b (x - L)f(x)dx}{\int_a^L (L - x)f(x)dx} = \frac{E[\max(x - L; 0)]}{E[\max(L - x; 0)]} = \frac{EG(L)}{EL(L)}$$

Which, as we can see, will result in the final simplified equation:

$$\Omega(L) = \frac{EG(L)}{EL(L)}$$

What illustrates the core of the Omega measure can be interpreted as EG(L), the numerator representing the expected value of excess gain (x-L), known as *Expected Gain* (EG), while the denominator is the expected value of loss (L-x), called Expected *Loss* (EL). This relationship can also be understood as "what one expects to gain if one wins compared to what one expects to lose if one loses".

The previous explanation and theoretical validation of the simplified formula of the Omega measurement by Keating and Shadwick (2002) made by Kazemi et el (2004) will be especially useful for the understanding of the OCAPM since this measurement plays a central role in the model.

The Omega CAPM or OCAPM model uses the theoretical framework developed by Markowitz (1952) and Sharpe (1964) together with the OPM (*Omega Performance Measure*) first presented by Keating and Shadwick (2002) and later refined by Kazemi et el (2004) in its composition, and aims to use the OPM instead of the mean-variance relationship of the previous models through the modification of their betas and had its first relevant literature written by Vasconcelos et al (2013).

Rescuing the equation postulated by Kazemi et el (2004) to OPM:

$$\Omega = \frac{\int_L^b (x-L) f_X(x) dx}{\int_a^L (L-x) f_X(x) dx} = \frac{EC}{ES} = \frac{E[\text{Max}(X-L;0)]}{E[\text{Max}\{L-X;0\}]}$$

And following the guidelines described by the model of Vasconcelos et al (2013) where all the detailed algebraic processes can be found and where he uses the premises that make up the CAPM model of Sharpe (1964) such as the understanding of the impossibility of zeroing systematic risks and the notion previously introduced by Markowitz (1952) of the possibility of reducing non-systematic risk through diversification,  it is possible to arrive at the final equation of the OCAPM model which can be found below:

$$E[R_i] = L + \beta_i (E[R_m] - r_f)$$

At first, it can be confused with the original formula of the CAPM, because its composition is identical, except for the composition of the Beta, because the OCAPM model was built from the Omega measure so that, through its use instead of the linear mean-variance relationship of the original model, it is possible, through the use of the Omega, to understand all the terms of the distribution of expected returns. Where βi is equal to:

$$\beta_i = E\left[\frac{(R_m - L)(R_i - L)}{|R_m - L|}\right] \frac{1}{E[|R_m - L|]}$$

Only a practical constraint should be respected, although it is not a theoretical constraint if Rm is equal to L resulting in 0 the Beta cannot be set, so the constraint that applies to the model is | Rm − L| must be other than 0. The interpretation of the Betas of the OCAPM model is the same as that of the CAPM model, although they do not represent the same relationship, since the Beta of the CAPM represents the relationship between covariance and variance, while in the OCAPM it also considers all distributions of returns above and below L.

## RECENT WORKS INVOLVING PORTFOLIO SELECTION USING EVOLUTIONARY ALGORITHMS

In this topic, some recent studies that used evolutionary algorithms for the selection of portfolios were listed. Next to the name of the article, a brief summary of the objectives and results presented will be attached.

Starting with the interesting paper by Khin Lwin et al (2014) in which the authors propose a different approach to the so-called Multiobjective Evolutionary Algorithms (MOEAs), with the aim of expanding the DEMO algorithm. This algorithm represents one of the recent approaches that combine the advantages of DE (R. Storn and K. Price) with the mechanisms of Pareto-based ordering and crowd distance sorting.

For the experiments, the authors evaluated the performance of four MOEAs: the Unmastered Genetic Classification Algorithm (NSGA-II), the Strength Pareto Evolutionary Algorithm (SPEA2), the Pareto Envelope-Based Selection Algorithm (PESA-II), and the Pareto Archived Evolution Strategy (PAES). The optimizations considered up to 1318 assets.

The operation of DEMO involves the maintenance of a population of individuals, where each one represents a possible solution to the optimization problem. During the process of evolution, DEMO allows the capacity of your population to be expanded to include newly found solutions that are not dominated by others. This makes it possible for these new, unmastered solutions to immediately participate in the generation of subsequent candidate solutions. This feature of DEMO promotes a rapid convergence towards the true Pareto frontier.

The authors, following the general scheme of DEMO, present a learning-driven multi-objective evolutionary algorithm called MODEwAwl (*Learning-Guided Multi-Objective Evolutionary Algorithm*). The main differences between MODEwAwl and conventional DEMO can be found in the article, along with the code, the proposed constraints, and the way they were implemented.

The results obtained with MODEwAwl are significant and promising. The authors highlight that MODEwAwl is not only capable of generating high-quality portfolios with additional constraints, but is also effective in resolving a reasonable number of assets (up to 1318). In addition, the proposed algorithm outperformed the four most widely used MOEAs: NSGA-II, SPEA2, PESA-II, and PAES.

Mishra et al (2016) present a new portfolio optimization model called PBMV (*Prediction Based Mean-Variance*), as an alternative to the traditional Markowitz mean-variance model, to solve the problem of portfolio optimization with constraints. In Markowitz's model, the error according to the authors is to use the average of past returns as an estimate of future returns. In PBMV, expected future returns are predicted with the use of a low-complexity artificial neural network. Portfolio optimization is then carried out through multi-objective evolutionary algorithms (MOEAs). In addition, a multi-objective optimization algorithm based on swarm intelligence called SR-MOPSO (*Self-Regulating Multiobjective Particle Swarm Optimization*) is proposed and employed to solve the problem.

The study compares the Pareto solutions obtained by the PBMV with those obtained by the Markowitz model and two other competitive MOEAs, using six performance metrics and the Pareto boundaries. A non-parametric statistical analysis is also performed to compare the performance of the algorithms in pairs. The results reveal that the approach based on the PBMV model offers better Pareto solutions while maintaining adequate diversity, and is comparable to the Markowitz model. Notably, the PBMV-based SR-MOPSO algorithm stands out as the best option in relation to the MOEAs evaluated. These results have significant implications for portfolio optimization with constraints and can be applied to other practical problems.

In addition, in addition to the 2 articles previously mentioned, the work done by Silva et al (2019) is relevant in view of the results presented and claimed by the authors, where in the work they address several variants of the medium-variance portfolio selection (PSP) problem through a unified multi-objective optimization (MO) approach using the *Adaptive Ranking Multi-Objective Particle Swarm Optimization* algorithm (ARMOPSUS). ARMOPSO introduced a classification procedure based on unmastered ordering, crowd distance, and a new mechanism called cost-effectiveness. This, according to the authors, is one of the first generic approaches to dealing with multiple variants of PSP in a single framework, according to an extensive review of the PSP literature over the past 20 years by the researchers involved.

ARMOPSO was tested on five variants of the medium-variance PSP, including typical financial market constraints such as minimum and maximum limits, cardinality, whole lot sizes, and pre-allocation. The results were compared using several specific metrics, such as spacing (S), generational distance (GD), diversity metric, hypervolume (HV), error (Er), mean return error (MRE), return error variance (VRE), as well as mean percentage error (MPE), minimum (MinPE), maximum (MaxPE) and median (MedPE).

The results of the extensive computational experiments demonstrated that ARMOPSO achieved a highly competitive performance in all variants and in most of the metrics evaluated, when compared to the specific methods for problems proposed in the literature. This highlights not only the efficiency of the unified method, but also its remarkable robustness.

The authors also point out that future work may involve the development of more effective ways to deal with the difficulty of practical application, helping the algorithm to find improved unmastered solutions and, consequently, a better quality frontier. Advancements in this regard can also reduce the CPU time spent by the procedure, especially for large instances. In addition, it is believed that ARMOPSO can be applied to solve other multi-objective problems.

## METHODOLOGY

### DATA COLLECTION

For this study, the American companies that make up the Dow Jones Index were chosen. The daily closing prices of the assets of each market were collected, and then the logarithm of the daily price change was calculated, preparing them for subsequent analyses. Companies that presented less than 30% of the prices in relation to the days collected were excluded from the sample.

The time windows were defined as follows:

Janela 1 (*in-sample*) 1994-1998, janela 2 (*out-of-sample*) 1999-2003.

Janela 3 (*in-sample*) 2004-2008, Janela 4 (*out-of-sample*) 2009-2013.

Janela 5 (*in-sample*) 2014-2018, Janela 6 (*out-of-sample*) 2019-2022.

### OPTIMIZATION APPROACHES

To carry out all the optimizations proposed by this work, the Spyder framework was used, which works through the Python programming language. The optimization proposals in question were: (i) CAPM (Sharpe), (ii) Omega CAPM (OCAPM), (iii) Multiobjective Evolutionary Algorithm 1 and 2

The following restrictions have been applied to all optimizations:

1- Cardinality constraint: corresponds to the maximum number of assets that the portfolio must optimize. The value used in the research was 8

2 – Budget constraint: refers to the sum of the weights of the assets being equal to 1

3 – Restriction of asset weights: The weight of each asset must be greater than or equal to 0

### CAPM Optimization

In the CAPM optimization, the algorithm developed by Sharpe was used, which has in its composition the objective of solving quadratic programming problems, finding the weights of the assets that maximize the objective function in the following image:

$$\text{Maximizar} \sum_i w_i r_i - \frac{1}{r_t} \sum_{i,j} w_i w_j \sigma_{ij}$$

Where Wi is the weights of the i-th asset in the portfolio, Ri is the expected return of the i-th asset, Rt is the investor's risk tolerance, and σij is the covariance between the returns of the assets.

It is still necessary to maximize the Sharpe ratio through the maximization problem described below:

$$\mathrm{SR} = \frac{R_p - R_f}{\sigma_p} = (R_p - R_f){\sigma_p}^{-1} = (\sum_i w_i r_i - R_f)(\sum_{i,j} w_i w_j \sigma_{ij})^{-0.5}$$

$$\text{sujeito a} \sum_{i=1}^{n} w_i = 1 \text{ e } w_i \geq 0$$

Where Rp is the return of the portfolio p, Rf is the Risk-Free Rate, σp is the standard deviation of the portfolio p, wi and wj are the weights of the assets i and j respectively, ri is the expected return of the i-th asset, and σij is the covariance between the assets i and j.

The maximization of the objective functions previously mentioned were done through computational experiments through the Spyder framework, which uses the Python programming language. The code was fed by the logarithm of the variation in the prices of capital market assets in both the US and Brazil and was made in a single-objective manner.

## OCAPM Optimization

To solve the issue of maximizing the Omega Index, which is considered a non-convex problem, it is necessary to solve a family of linear programs or a single fractional linear program, thus delimiting the traveled area, solving the problem of non-convexity. The code referring to the aforementioned maximization was made in Spyder based on the Scipy.optimize package in the Python language. The optimization of the OCAPM was also done in a single-objective way.

The formula for the objective function of the Omega Index can be seen below:

$$\text{Maximizar} \quad \text{Índice Ômega} = \frac{EG_p(L)}{EL_p(L)},$$

$$\text{sujeito a} \sum_{i=1}^{n} w_i = 1 \text{ e } w_i \geq 0$$

Where EGp (L) is the *Expected Gain*, ELp (L) is the *Expected Loss* both refer to the portfolio, L is a threshold, Wi is the weights of the assets in the portfolio i, and n is the number of assets in the portfolio.

## Multi-Objective Evolutionary Optimization 1 and 2

Multi-objective optimization consists of several objective functions being executed simultaneously in order to maximize or minimize them, subject to a set of constraints that the solutions must satisfy. To perform this optimization, the general form of a multiobjective optimization problem conceptualized by Deb (2001) was used and can be seen below:

$$\begin{aligned} \text{Maximizar ou minimizar} \quad & f_m(x) \quad m = 1, \ldots, N_{obj} \\ \text{sujeito a:} \quad & \\ & g_j(x) \geq 0 \quad j = 1, \ldots, J \\ & h_k(x) = 0 \quad k = 1, \ldots, K \\ & x_{lb} \leq x \leq x_{ub} \quad i = 1, \ldots, N_{var} \end{aligned}$$

Where:

x is a vector with N decision variables

Nobj is the number of optimization goals

gj(x) are the constraints of inequality

hk(x) are the equality constraints

J and K are the numbers of inequality and equality constraints

XLB and XUB are the lower and upper bounds of each decision variable

Knowing that it is not possible to simultaneously optimize all objectives, the solution used to mediate this selection was the use of Evolutionary Algorithms (AE), which use the value of the objective functions of each individual's problem to, through classification tests, choose which ones will remain for the next interactions. GAs work by coding candidate solutions to a problem through the use of genetic operators, i.e., through the concepts of Darwinian evolution, mutation, and crossbreeding. In the present study, the NSGAII, NSGAIII, IBEA and GDE3 algorithms were used.

Based on these concepts, the multiobjective optimization used as central attributes the maximization of the Omega measure, the minimization of the *maximum drawdown*, and the conditional correlation of the VIX (CVIX). Like all the optimizations present in this work, both were also performed through the Spyder framework, based on the Python programming language.

## RESULTS AND DISCUSSION

For the evaluation of the results, the accumulated returns outside the sample of the optimized portfolios, i.e., *out-of-sample*, were considered, these being the samples that represent the performance of the distribution of the weights of the assets in the portfolio obtained through the *in-sample optimizations*. Thus, it is possible to evaluate the performance of the portfolio obtained in the optimization if it were implemented by an investor in practice in the years that comprise the *out-of-sample period*.

At first, it is important to point out that due to the convexity of the optimized Sharpe, Omega portfolios, they will converge in a single solution and have a single point. With regard to the evolutionary algorithms, the choice was made among the 50 best executions of the algorithm, for this reason the unmastered portfolios that represent more than one portfolio and do not have dominance between them were also considered, so aiming at the best understanding we chose to represent everything in the form of a *boxplot*.

Contextualizing the presentation of the results and their analysis, it is important to highlight the 6 metrics for analyzing the data obtained through optimizations. First, due to the nature of the optimization made by the algorithms used, i.e., in a multiobjective way, it is not possible to compare the solutions obtained based on the values of their objective functions directly, because the result of the optimization is a frontier of optimal solutions.

Thus, it is necessary that the first evaluation metric be the calculation of the hypervolume of the results presented by the algorithms used (NSGAII, NSGAIII, IBEA and GDE3). This evaluation metric seeks to evaluate the quality of the frontier obtained by weighting two central characteristics: The proximity of the Optimal Pareto frontier and the extent to which the solutions are well distributed along the frontier in order to identify the solution that best covers the frontier, so the first characteristic concerns convergence and the second refers to diversity.

The hypervolume metric was defined by Zitzler and Thiele (1998) and can be obtained by considering: **S** as the set of undominated solutions generated by a multiobjective algorithm, **R** as a reference point, dominated by all solutions of **S,** and **vi** the hypercube formed from the space dominated by the Si∈S solution having the values of **R** as a limit. Once the hypervolume-based metric was obtained, as explained above, it was possible to identify the algorithm that presented the best set of optimal solutions. From the choice of the algorithm that presented the best performance, 5 more metrics were applied, 3 of them aiming to measure the risk associated with the portfolio formed (CVaR, maximum Drawdown and Coefficient of variation), one related to the return obtained in the *out-of-sample window* (Accumulated Return) and the last one measures the percentage of allocation in each asset chosen by the evolutionary algorithm previously chosen by the analysis of the hypervolume metric. At the end of the calculation of the chosen and previously mentioned metrics, it was possible to obtain the graphs, separated by time window and by evaluated market. All the mentioned metrics were applied both in the algorithm chosen as the most efficient in terms of the Sharpe and Omega ratios and also in the Index of the evaluated market, except for the hypervolume metric, because due to the nature of the indexes addressed, it is not necessary, since they present a single solution.

Before presenting the graphs obtained through the concepts mentioned above, it is interesting to contextualize the 3 risk metrics used for a better understanding of the analyses made, apart from CVaR, maximum drawdown and coefficient of variation.

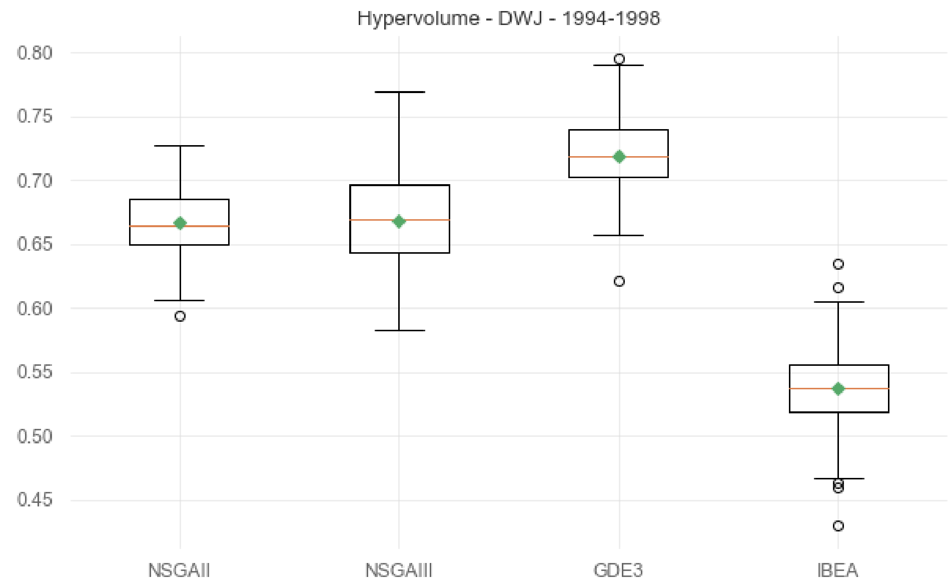
The CVaR, which stands for *Conditional Value at Risk* , can also be called Average Expected Loss or *Expected Shortfall* and aims to answer the question "What is the average loss incurred by the portfolio in x% worst-case scenarios?"  That is, it quantifies the average size of the loss in the worst possible scenarios of the portfolio, and in the context used in the analysis of this article it will measure the average loss of the worst days of the portfolio formed, thus having the ability to detect the presence of catastrophic events. For its use, it is necessary to first define the level of reliability used in the metric, for example, a 95% confidence level means that the point of interest of the potential losses has been defined for 5% of the worst possible scenarios, thus being possible to calculate the average of the potential losses associated with these worst scenarios present in the chosen portfolio. The last example describes the level of reliability defined for the analyses of this study, so the CVaR will tell us the average loss of the 5% worst days of the portfolio being analyzed.

The maximum *drawdown*, on the other hand, is the percentage difference between tops and bottoms, thus being able to indicate what is the maximum "pain" point present in the portfolio, that is, what is the largest possible loss stage present in the portfolio formed during the defined period. It can also be understood as the largest percentage reduction in the value of an investment or portfolio, measured from its previous peak to its subsequent lowest point.

And finally, the coefficient of variation, which is used to measure the risk/return ratio of the portfolio, i.e. how many units of risk will be added for each unit of return. Therefore, it is interesting to always look for the lowest possible coefficients of variation, as this will indicate a better relationship between these two variables. It can be obtained by dividing the standard deviation of the returns by the expected value of the returns.

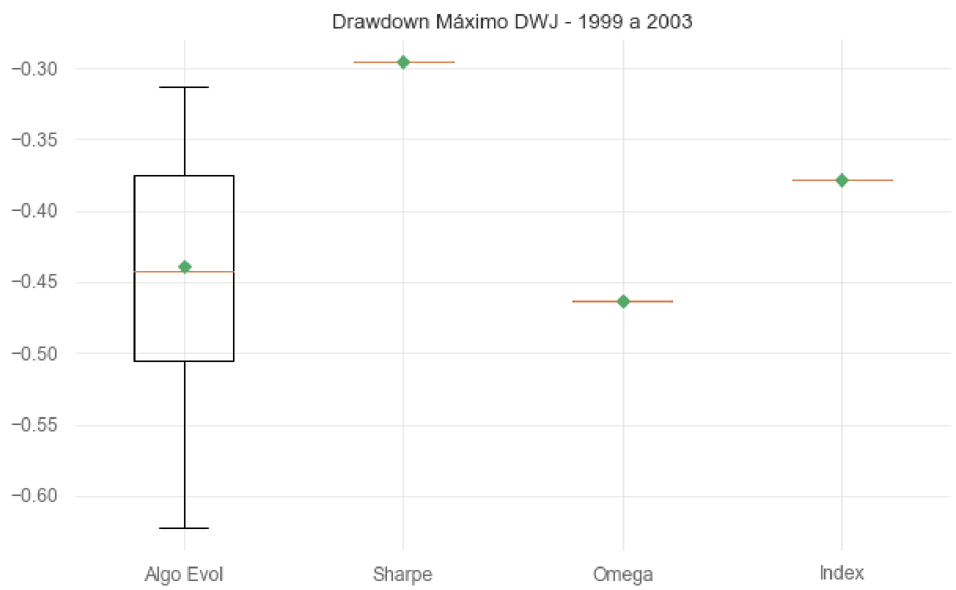
From here, the analyses were presented according to the market in which the optimization was made, as well as its previously defined time window.

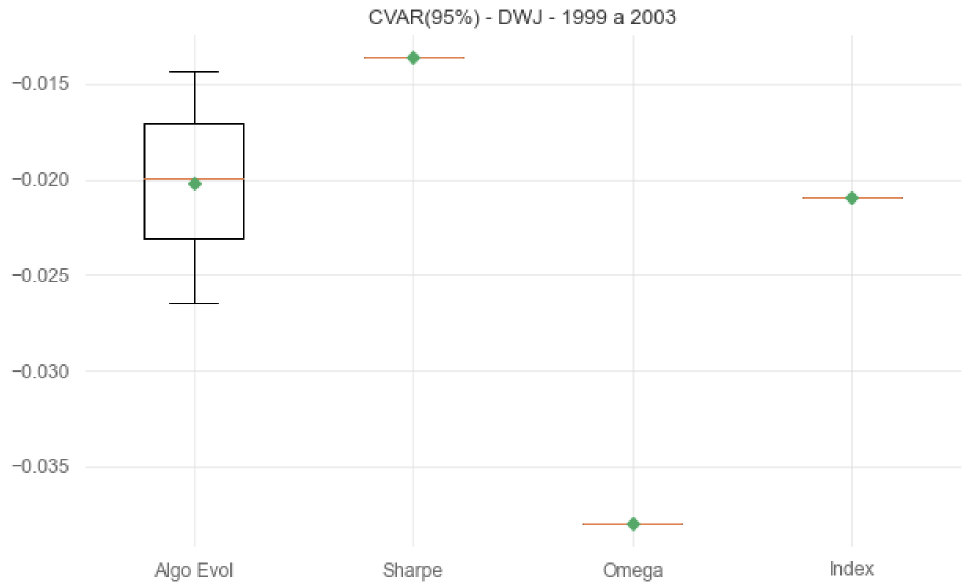
Window 1 – Dow Jones Companies – *Out-of-Sample* (1999-2003)

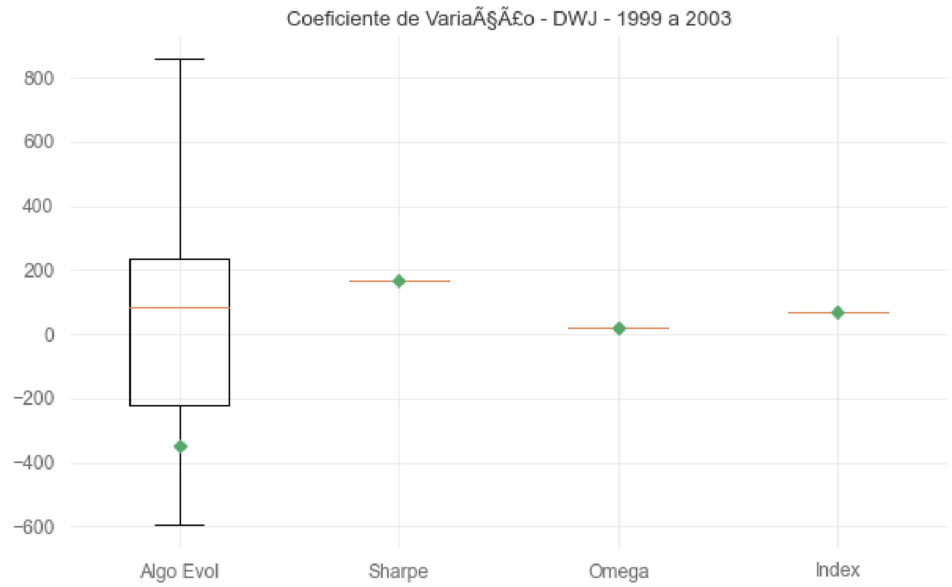



As can be seen, the algorithm that presented the best performance based on the hypervolume metric was GDE3, and therefore it was the algorithm chosen to make the comparisons with Sharpe, Omega and the Index of the market in question. Therefore, within that section, the legend "Algo Evol" on the charts is representing GDE3's performance.

The following are the 3 graphs referring to the risk metrics used:





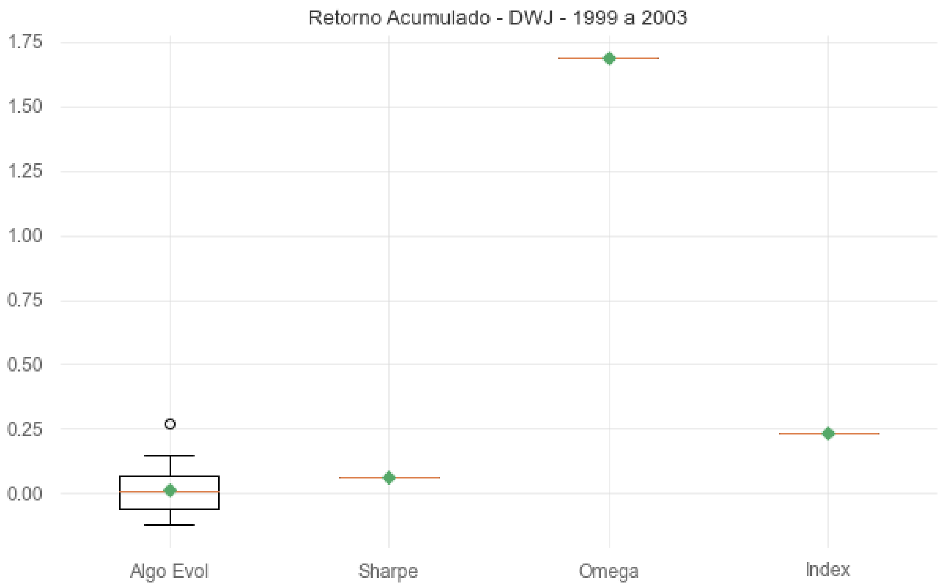




Regarding the risk metrics in the first window of the DWJ market, we can observe that the evolutionary algorithm together with the Omega measure presented the highest *maximum drawdowns* among the portfolios evaluated, representing a reduction of up to 45% of the initial capital along the way. It is also possible to lose an average of 2% on the worst days for the algorithm.

The portfolio based on the Sharpe ratio was the one that presented the best risk metrics in this window, but just like the evolutionary algorithm presented a cumulative return very close to 0, which ends up not making much difference to have good risk metrics in this case, as it proved to be inferior even to the Index of the evaluated market.

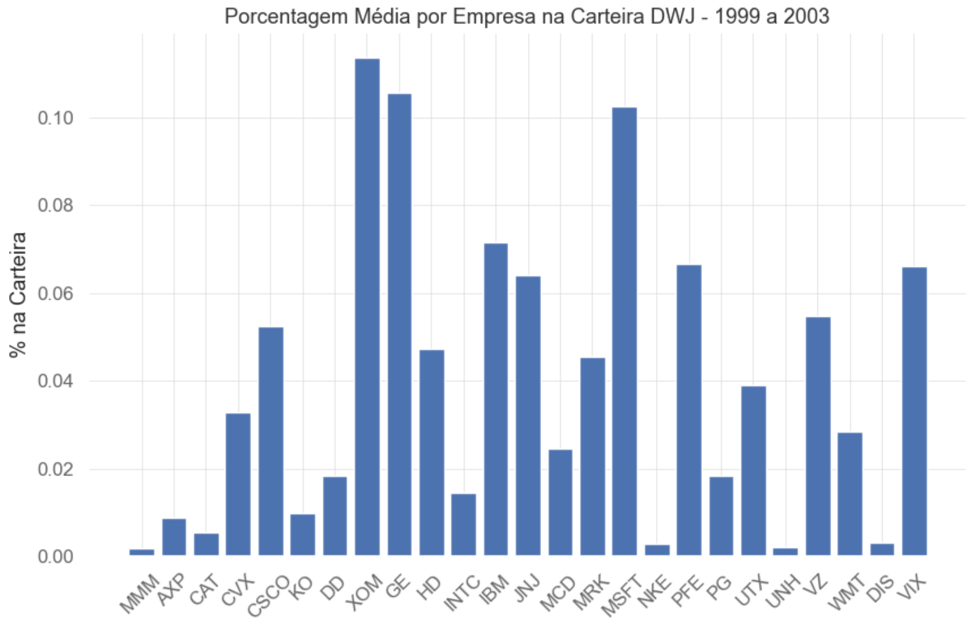
In addition, it is interesting to note that the Omega measure presented a huge associated risk, being able to lose an average of 4% on the worst days and more than 45% at some point in the time window, represented by *the drawdown* and this can be explained by the low diversification applied by the optimization made with the Omega measure, where the choice was to allocate 100% of the capital in a single company under the acronym UNH (UnitedHealth Group Inc).

On the other hand, the evolutionary algorithm, despite having diversified much more than the Omega measure, was not able to mitigate the associated risks just by diversifying more, presenting results related to risk very similar to the Omega measure, but when it comes to the accumulated return it performed much lower, losing even to the DWJ Index (Characteristic that was only observed in the American market, where the Index due to the maturity of the market represents a viable allocation option, which does not happen in the Brazilian market, this concept was better worked in the IBOV time window)



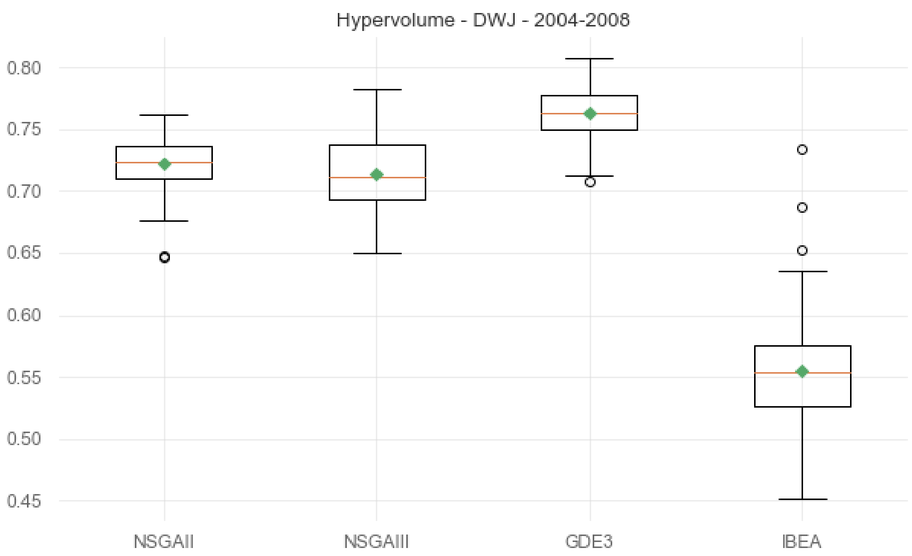

As you can see in the chart above, the cumulative return of the evolutionary algorithm (GDE3) was very close to 0, as well as Sharpe, where both lose even to the Index in this first window of the DWJ market. With emphasis on the Omega measure, which presented a return close to 175%, however, as previously mentioned, the optimization based on this measure opted for the total allocation of capital in only 1 asset. And as has been demonstrated by the risk metrics presented, this represents a significant increase in the risk associated with the portfolio formed.

Next, the choices made by the evolutionary algorithm chosen as the best among the 4 evaluated can be visualized. The capital-allocation weights on each DWJ asset along with the allocation to the VIX, which is made on the basis of the CVIX. And although the VIX is not an asset in itself, we chose it because it made the algorithm interpret such an index as one, so in times of great volatility in the market, it can choose to allocate a percentage of the available capital to this imaginary asset, as a form of protection, so that in practice the capital allocated to the VIX is actually out of the market and protected from its variation. This logic was applied to all other windows.



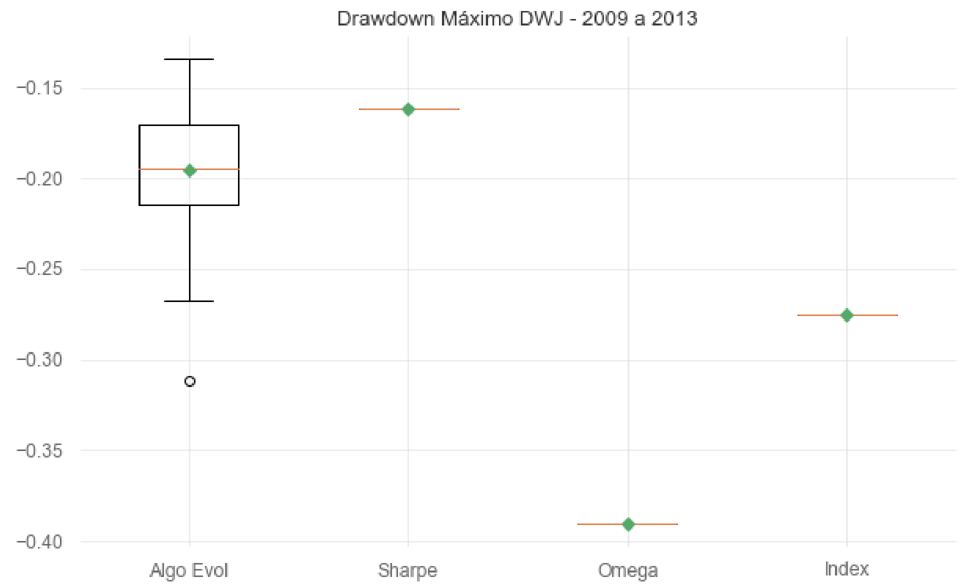

Due to the return close to 0 and below even the market index, this chart has no practical use in itself in the 1999-2003 time window.

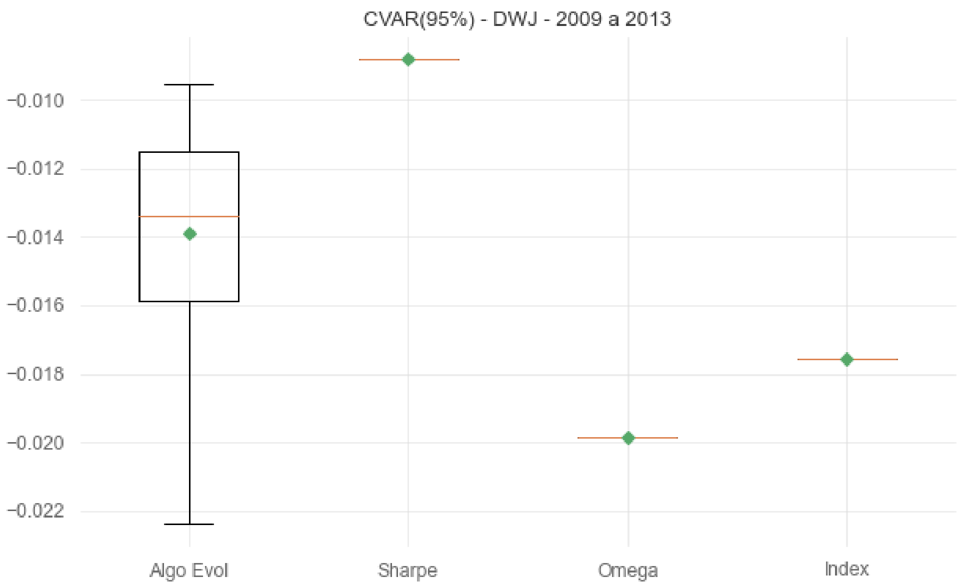
Window 2 – Dow Jones Companies – *Out-of-Sample* (2009 - 2013)

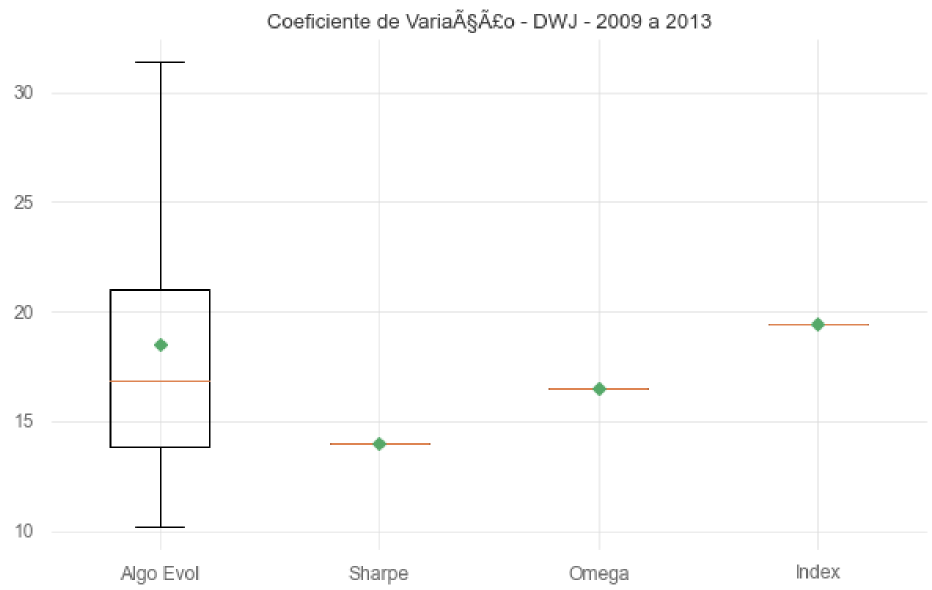


As in the previous windows, and in the subsequent windows, the winning algorithm within the hypervolume metric was GDE3 and so from here on the hypervolume graph needs no legend.

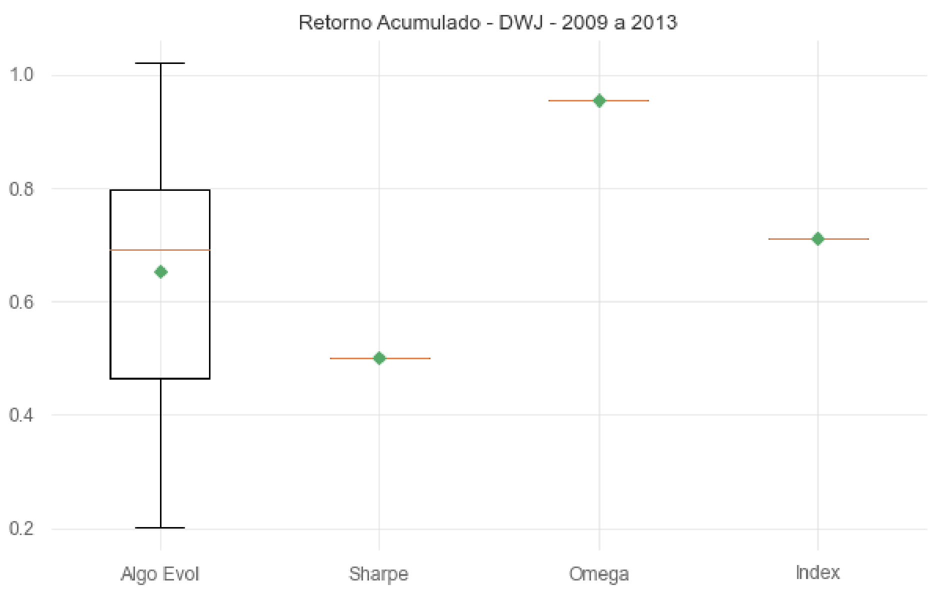
Risk metrics:



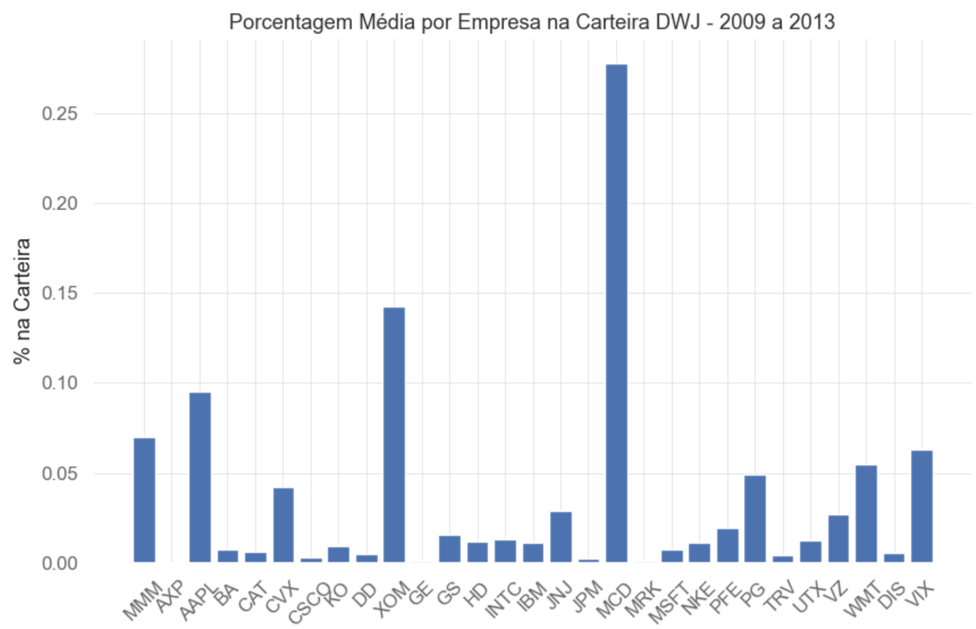






In this window a certain pattern repeats itself again, Sharpe presents the best risk metrics, along with the evolutionary algorithm in this specific window. Another thing that can also be observed and has already been discussed in previous windows and ends up being seen in this window is the optimization based on Omega presenting the worst risk metrics, this time accompanied by the market Index, even though it is the most mature market Index, the American one.

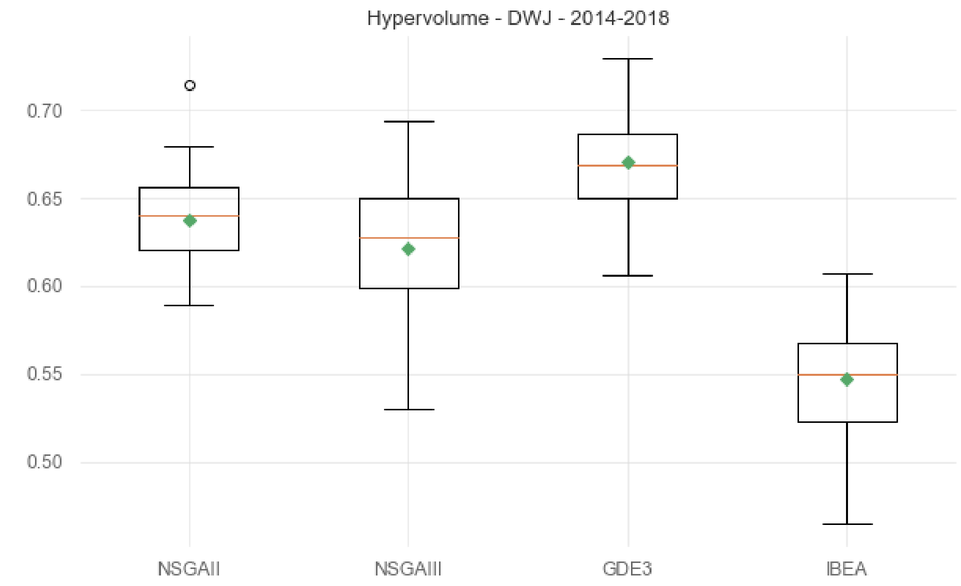



As for the accumulated returns, one more pattern can be identified, Omega presents the highest return by far compared to the other optimizations and in relation to the Market Index, which on this specific occasion obtained a very satisfactory return, reaching a higher than Sharpe and the evolutionary algorithm itself. However, as in previous windows, the higher premium offered by Omega is directly related to a considerable increase in portfolio risk, which reforms the idea that despite the higher yield, it becomes more interesting to opt for Sharpe or even in this case for the evolutionary algorithm, which has shown to deal well with risk in this window. and was able to deliver a cumulative return of approximately 70%, even higher than Sharpe. The 70% performance of the Index in this time window can be explained by the global macroeconomic moment of the time, where after the 2008 crisis assets in general were "cheap" and consequently with the recovery of the economy in subsequent years offered a return above normal. As in previous windows, Omega has low diversification, but this time choosing to allocate 9% to the VIX and obtaining considerable returns despite not so interesting risk metrics. Sharpe, in addition to being more diversified, also chose to allocate about 10.5% of the capital to the VIX.

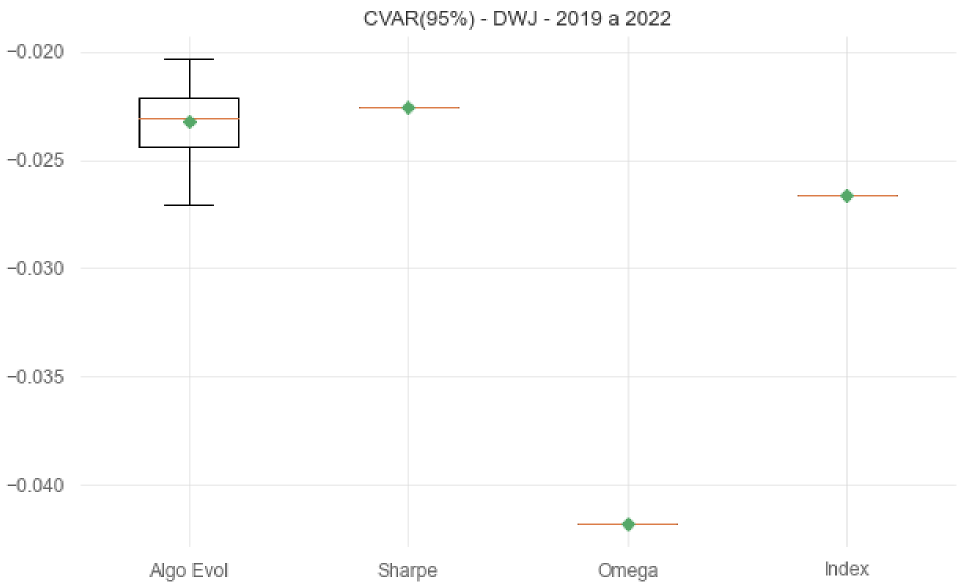


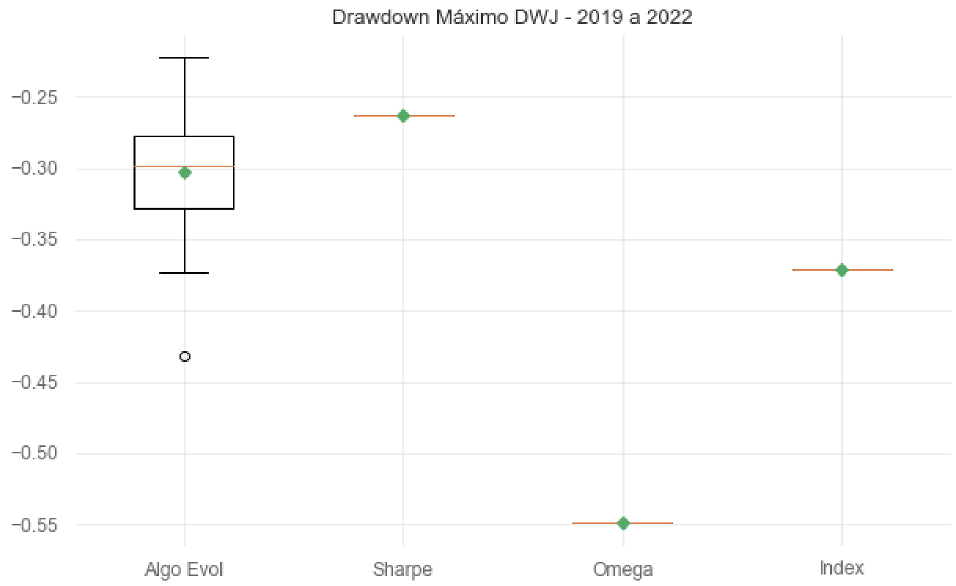

The choices made by the evolutionary algorithm are represented above, it is interesting to highlight the allocation of more than 5% in the capital, also in the VIX.

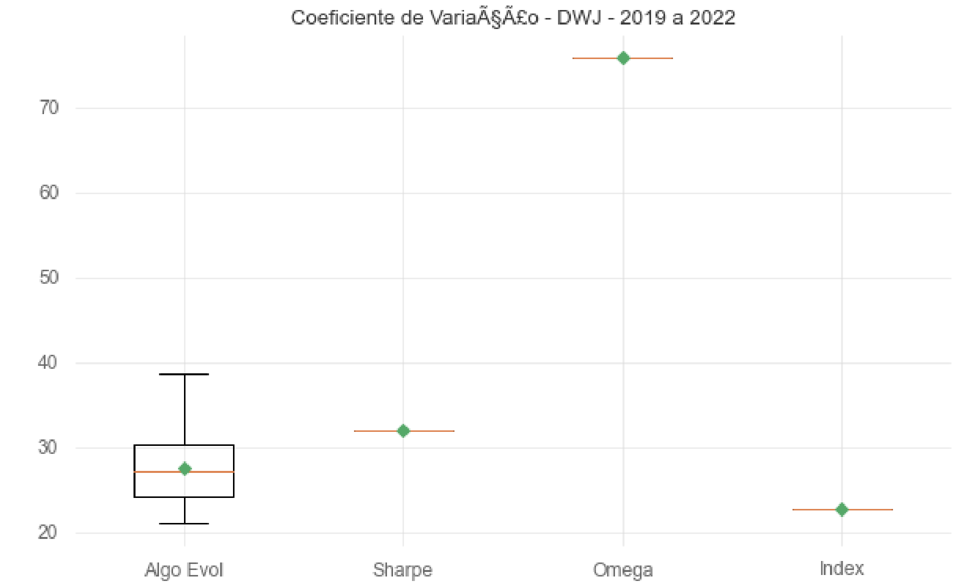
Window 3 – Dow Jones Companies – *Out-of-Sample* (2019 - 2022)



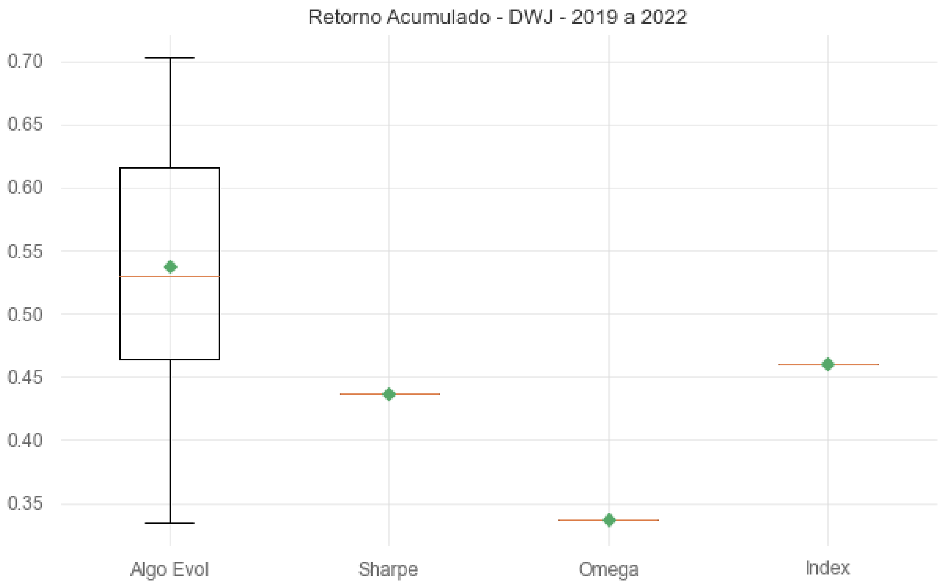

Risk metrics:



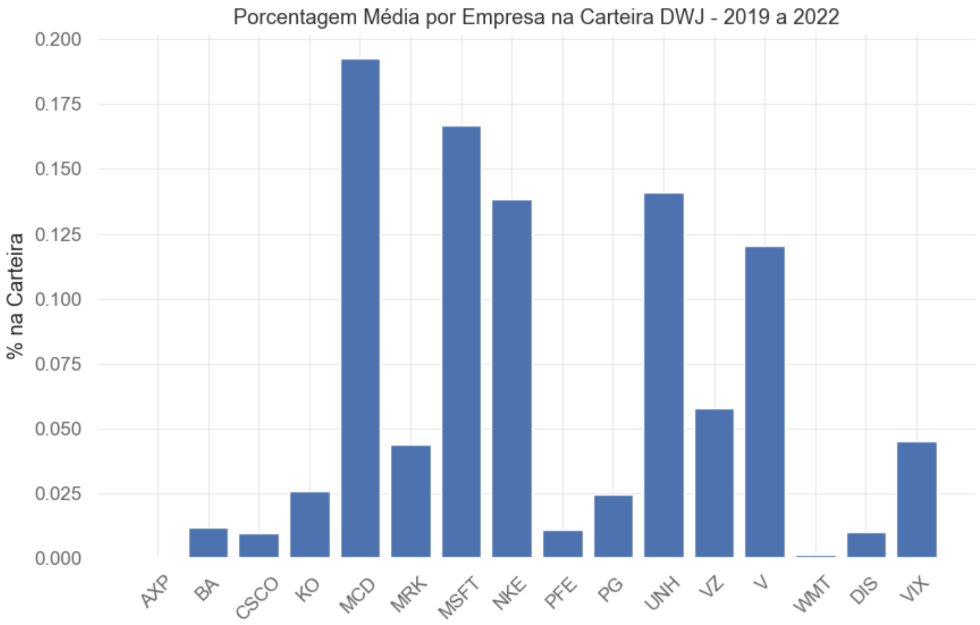



In the last window of the American market the results do not diverge from the analyses made in the previous two, once again the optimization made by Omega presents the worst risk metrics, with the Index having shown us a good performance as the risk metrics following the same pattern of the Index performing well in the more mature market of DWJ. Sharpe remains the winner in terms of risk-reward ratio, followed by the evolutionary algorithm. Making it so that once again the rational choice of the investor focuses on the choice between these two portfolios.




In this window, the risk associated with the choice for low diversification made by the Omega index is evident, which this time presented a cumulative return lower even than the Index of the American market. With emphasis on the evolutionary algorithm that presented the best accumulated return combined with good performance in relation to risk metrics, gaining considerably in accumulated return even from the Sharpe ratio, which had been presenting relatively better returns in relation to the associated risk until then. Omega chose to allocate 86% of the capital to only one company (GE), followed by 7% to IBM and 6% to the VIX, and even then presented accumulated returns lower than the index, indicating high exposure to risk and low accumulated return, unlike Sharpe, who again presented high diversification with 7% allocation to the VIX. The choices made by the evolutionary algorithm can be seen below:




It is important to highlight that the greater the diversification, not only in relation to the number of assets, but also in relation to the different sectors of the economy in which these chosen companies operate, the lower the coefficient of variation. This explains the high coefficients presented by the Omega measure. It is also important to point out that in all the optimizations made by the evolutionary algorithm there was part of the capital allocated within the VIX.

## CONCLUSION

The objective of this work was to perform optimizations with purely convex attributes in a mono-objective way and then to compare the expected returns of the assets outside the sample of these optimizations with the results from the optimizations made in a multiobjective way with non-convex attributes, with emphasis on CVIX and *drawdown* maximum. Thus, it is possible, through the evaluation of the results, to know if the portfolios formed exceed the theoretical market portfolios suggested by the CAPM model. The multi-objective optimizations mediated by evolutionary algorithms showed us a better performance than the Sharpe and Omega portfolios with higher out-of-sample returns, as well as a better mean-variance ratio. The results then confirmed that the use of only the mean and variance in the construction of optimized portfolios should not be the only attributes to be considered, in particular this work demonstrates that the addition of an antifragile attribute as well as the maximum drawdown *in the portfolio optimization models can potentially increase the performance of the formed portfolio.*

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