A look at the composition of linear transformations in the language of matrices, some types of matrices of order 2 represented geometrically in the R2 plane

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ABSTRACT

The purpose of this note is to present compositions of linear transformations in the language of matrices, as well as to present geometric interpretations for some particular cases of order 2 matrices such as reflections around the x and y axes, reflections around the origin, contraction, expansion or homothetic, horizontal and vertical shear, counterclockwise rotation, orthogonal projection of u = (x, y) on the line G : y = ax, a /= 0, as well as the reflection of the same vector around this same line.
G. It is worth mentioning that your compositions in the language of matrices is a first model of computer graphics. Illustratively, for example, the expansion of factor k : H_k (x, y) = (kx, ky) or in the language of matrices, represents a computer zoom by zooming if k > 1 or contracting if 0 < k < 1.

Keywords: Teacher Training, Educational Technology, Artificial Intelligence, Digital Literacy.

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INTRODUCTION

A BIT OF HISTORY SURROUNDING THE NAME MATRIZ

It was only a little more than 150 years ago that matrices had their importance detected and came out of the shadow of the determinants. The first to give them a name seems to have been Cauchy, 1826: tableau (table). The name matrix only came with James Joseph Sylvester, 1850. His friend Cayley, with his famous Memoir on the Theory of Matrices, 1858, publicized this name and began to demonstrate its usefulness. Why did Sylvester give the name matrix to the matrices? He used the colloquial meaning of the word matrix, that is: place where something is generated or created. In fact, he saw them as "... a rectangular block of thermoses... which does not represent a determinant, but is as if it were a matrix from which we can form several systems of determinants, by fixing a number p and choosing at will p rows and p columns..."article published in the Philosophical Magazine of 1850, p. 363-370). Note that Sylvester still saw matrices as a mere ingredient of determinants. It is only with Cayley that they begin to take on a life of their own and gradually begin to supplant the determinants of importance.

EMERGENCE OF THE FIRST RESULTS OF MATRIX THEORY

It is often said that in a more advanced course of Matrix Theory - or its more abstract version, Linear Algebra - it should go at least up to the Spectral Theorem. Well, this theorem and a whole host of ancillary results were already known before Cayley began to study matrices as a remarkable class of mathematical objects. How can this be explained? These results, as well as most of the basic results of Matrix Theory, were discovered when mathematicians of the fifteenth and nineteenth centuries began to investigate the theory of quadratic forms. Today, we consider it essential to study these forms through notation and matrix methodology, but at that time they were treated scalarly. Let us show here the representation of a quadratic form of two variables, both via scalar notation and with the more modern matrix notation:

$$q(x,y) = ax^2 + 2bxy + sy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = X^t A X,$$

where: $X^t = \begin{pmatrix} x & y \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ e $A = \begin{pmatrix} a & b \\ b & s \end{pmatrix}$ $A^t = A$ (symmetric matrix)

The first implicit use of the notion of matrix occurred when Lagrange 1790 reduced the characterization of the maxima and minima of a real function of several variables to the study of the sign of the quadratic form associated with the matrix of the second derivatives of this function. Always working in a scalar way, he came to a conclusion that we now express in terms of a defined positive matrix. After Lagrange, in the nineteenth century, the Theory of Quadratic Forms became one of the most important subjects in terms of research, especially with regard to the study of its invariants. These investigations had as a by-product the discovery of a large number of results and basic concepts of matrices. Thus, we can say that the Theory of Matrices had as its motto the Theory of Quadratic Forms, since its basic methods and results were generated there. Today, however, the study of quadratic forms is a mere chapter in matrix theory. It should also be noted that the determinants contributed nothing to the development of the Matrix Theory.

COMPOSITIONS OF LINEAR TRANSFORMATIONS

Definition: Let you and B be vector spaces over R and let T be : U →B and G : B →W linear transformations. The compound G◦ T : U → W, is given by:

$$(G \circ T)(u) = G[T(u)], 6u \in U.$$

$$\begin{array}{ccc} U & \overset{T}{\rightarrow} & B \\ G \circ T & \diagdown & MG \\ & & W \end{array}$$

For the compound G ◦ T we have: the image of T is contained in or equal to the domain of G :

$$Fm(T) \subseteq D(G).$$

Similarly, we have:

For the composite T ◦ G to exist, it only makes sense when: the image of G is contained in or equal to the domain of G :

$$Fm(G) \subseteq D(T).$$

Next, the theorem that characterizes that composed of linear transformations are also linear.

Theorem : Let T ∈ 5 (U; B) and G ∈ 5 (B; W), then, G ◦ T : U → W, G ◦ T ∈ 5 (U; W) is linear.

Note:

5 (U; B) is the space of all linear transformations from U to B.

5 (B; W) is the space of all linear transformations from B to W.

Demo

Let G and T be linear transformations, then we want to show that:

$$G \circ T : U \rightarrow W$$

it's linear.

Indeed

$$\begin{array}{ccc} U & \overset{T}{\rightarrow} & B \\ G \circ T & \searrow & \downarrow G \\ & & W \end{array}$$

(i) $6u_1, u_2 \in A$, one has:

$$\begin{aligned}(G \circ T)(u_1 + u_2) &= G[T(u_1 + u_2)] = G[T(u_1) + T(u_2)] \\ &= G[T(u_1)] + G[T(u_2)] \\ &= (G \circ T)(u_1) + (G \circ T)(u_2).\end{aligned}$$

(ii) $6\lambda \in R$, $6u_1 \in A$, you get:

$$\begin{aligned}(G \circ T)(\lambda u_1) &= G[T(\lambda u_1)] = G[\lambda T u_1)] = \lambda G[T(u_1)] \\ &= \lambda (G \circ T)(u_1).\end{aligned}$$

Therefore, $G \circ T : U \rightarrow W$ is linear.

Examples:

1. Let $T : R^3 \rightarrow R^2$ and $G : R^2 \rightarrow R^3$ be linear transformations defined by:

(e) $T(x, e, x) = (s + e, s + 2x)$ (ii) $G(x, y) = (x + y, 2y, x - y)$

Pede-se: (i) $G \circ T$ (ii) $T \circ G$

Solution

(i) $G \circ T$

$$\begin{array}{ccc} R^3 & \xrightarrow{T} & R^2 \\ G \circ T & \searrow & \downarrow G \\ & & R^3. \end{array}$$

Before we start operationalizing, it is essential to understand the stickers they represent; Not the letters used, let's see:

G tells us that: it takes the sum of the first two coordinates of the domain, it is the first coordinate of the image, the fold of the second coordinate of the domain is the second coordinate of the image, and finally, the difference of the two coordinates of the domain is the third of the image. Symbolically, we have:

$$G(A, Q) = (A + Q, 2Q, A - Q)$$

Thus, it is easy to obtain the compound, namely:

$$\begin{aligned} (G \circ T)(x, y, x) &= G[T(x, y, x)] \\ &= G(x + y, x + 2x) \\ &= (x + y + (x + 2x), 2(x + 2x), x + y - (x + 2x)) \\ &= (2x + y + 2x, 2x + 4x, y - 2x). \end{aligned}$$

A very interesting way, and usually not in the texts in general, is to make the composition using matrix form in the canonical base for simplicity

$$[G \circ T] = [G].[T]$$

Constructing the corresponding matrices [G] and [T], we obtain:

$$[G] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \qquad (1)$$

and

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}. \qquad (2)$$

Hence, multiplying (1) and (2), it follows that:

$$[G \circ T] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix}.$$

By rewriting the matrix form G ◦ T in the canonical base, we get:

$$(G \circ T).\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ x \end{bmatrix}$$

Proceeding in a similar way, it comes:

(i) $T(x, y, x) = (x + y, x + 2x)$ (ii) $G(x, y) = (x + y, 2y, x - y)$

(ii) $T \circ G$

$$\begin{array}{ccc} R^2 & \overset{G}{\rightarrow} & R^3 \\ T \circ G & \searrow & MT \\ & & R^2. \end{array}$$

A brief comment, it is essential to understand the stickers they represent; Not the letters used, let's see:

T tells us: that: takes the sum of the first two coordinates of the domain, in the first coordinate of the image, the first coordinate of the domain with the fold of the third is the second coordinate of the image. In symbol, we have:

$$T(\omega, A, Q) = (\omega + A, \omega + 2Q).$$

Thus, it is easy to obtain the compound, namely:

$$
\begin{aligned}
(T \circ G)(x, y) &= T\,[G(x, y)] \\
&= T\,(x + y, 2y, x - y) \\
&= (x + y + 2y, x + y + 2\,(x - y)) \\
&= (x + 3y, 3x - y)\,.
\end{aligned}
$$

It is noteworthy that: usually does not appear in the texts in general, the composition using matrix form in the canonical basis

$$[T \circ G] = [T]\,.\,[G]\,.$$

Constructing the corresponding matrices [G] and [T], we obtain:

$$[G] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$

and

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Therefore, it comes:

$$[T \circ G] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

Rewriting in matrix form on the canonical basis, we get:

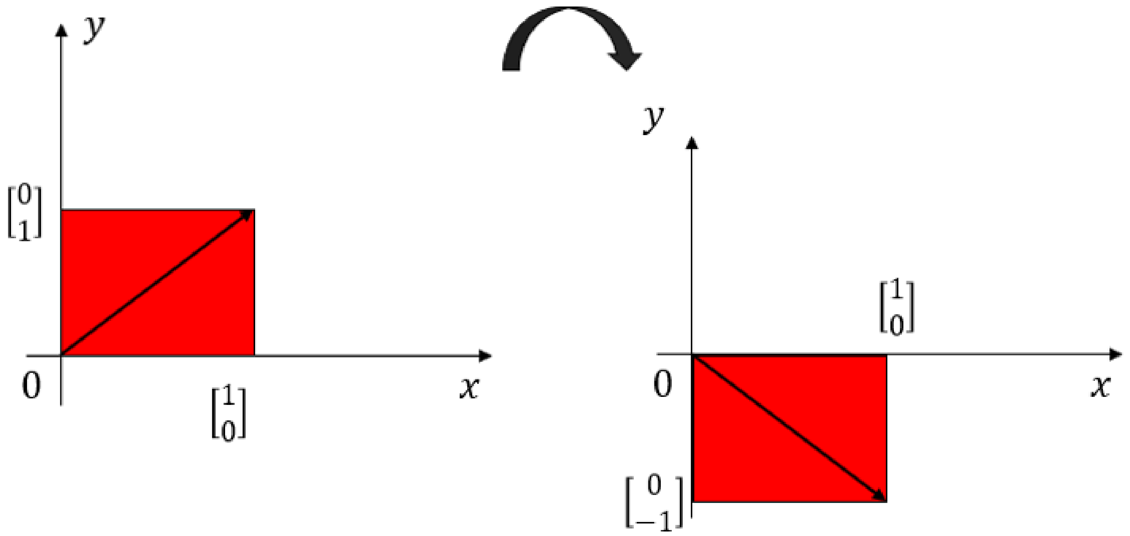
$$(G \circ T)\, x = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} x$$

TRANSFORMATION OF THE PLAN INTO THE PLAN

REFLECTION AROUND THE X-AXIS:

$$R_x : R^2 \longrightarrow R^2$$

$$(x, y) \longrightarrow R_x(x, y) = (x, -y)$$



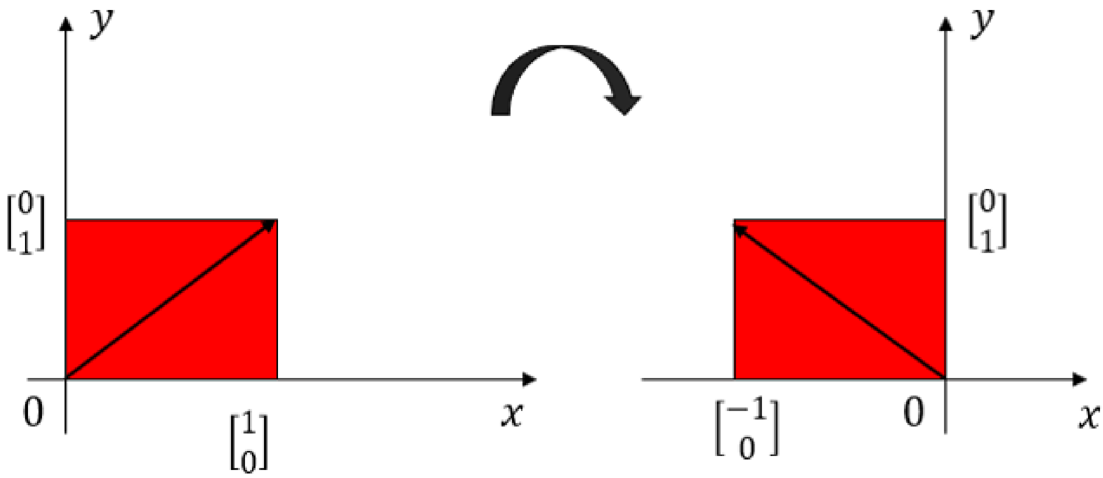
In the language of matrices, the reflection around the x-axis, described in matrix form, we have:

x — y

Reflection around the y-axis:

$$R_y : R^2 \longrightarrow R^2$$

$$(x, y) \longrightarrow R_x(x, y) = (-x, y)$$



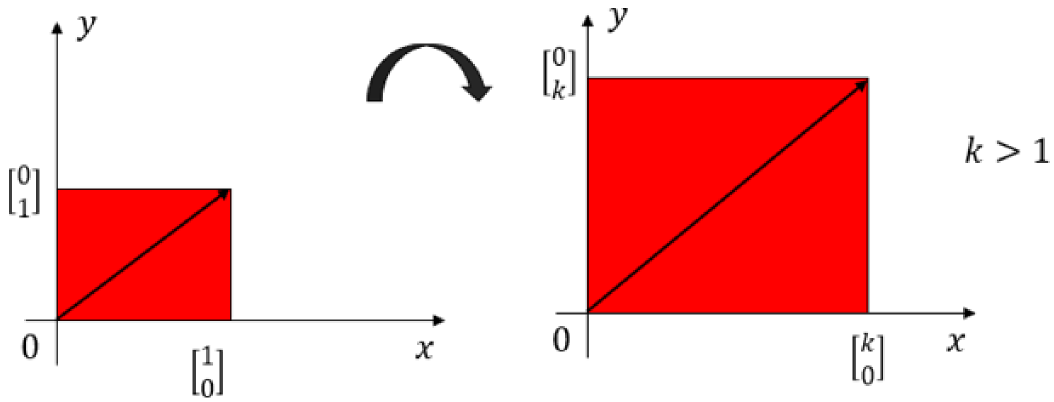
In the language of the matrices, the reflection around the origin will be given by:

$$R_o\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Homothetia, Contraction or Expansion

$$H_k : R^2 \longrightarrow R^2$$

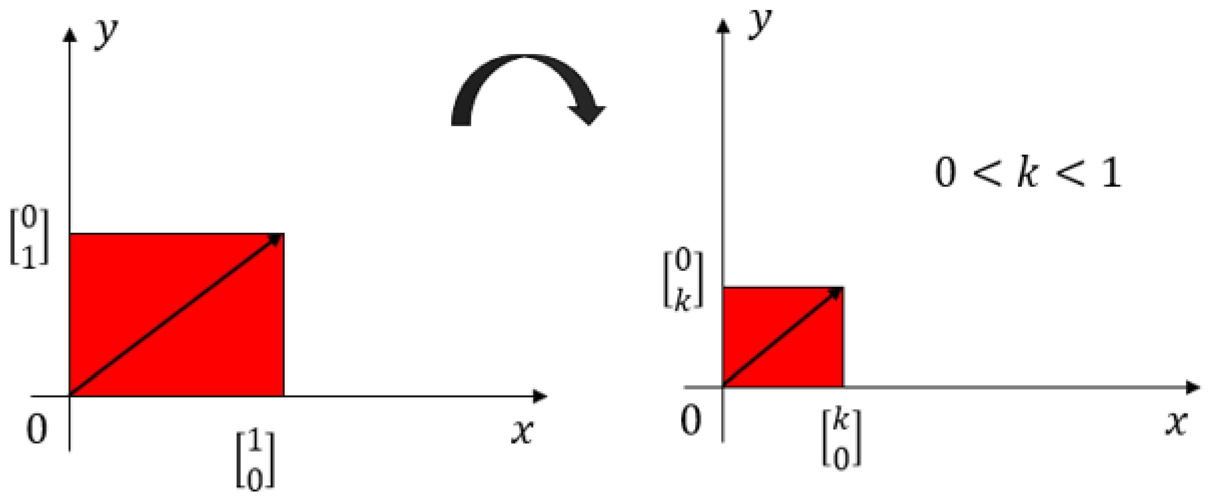
$$(x, y) \longrightarrow H_k\,(x, y) = (hx, hy)$$



In the language of the matrices, Homothetia, Contraction or Expansion, is given by:

’ x ’ ’ h 0 ’ ’ x ’
k ‚ ‚ ‚ ‚ ‚ ‚
0 h

IN AN ENTIRELY ANALOGOUS WAY, HOMOTHETIC, CONTRACTION OR EXPANSION, FOR 0 < H < 1, WE HAVE



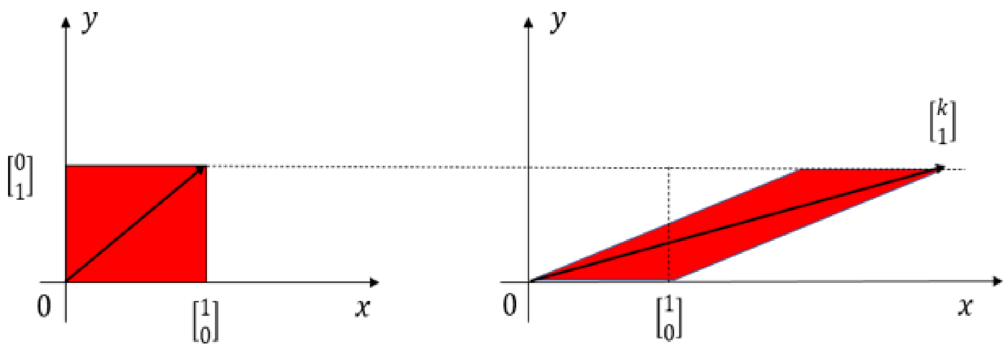
In the language of the matrices, Homothetia, Contraction or Expansion, is given by:

’ x ’ ’ h 0 ’ ’ x ’
k ‚ ‚ ‚ 0 h ‚ ‚ ‚

Horizontal shear of h-factor :

$C_k : R^2 \longrightarrow R^2$

$(x, y) \longrightarrow C_k (x, y) = (x + hy, y)$



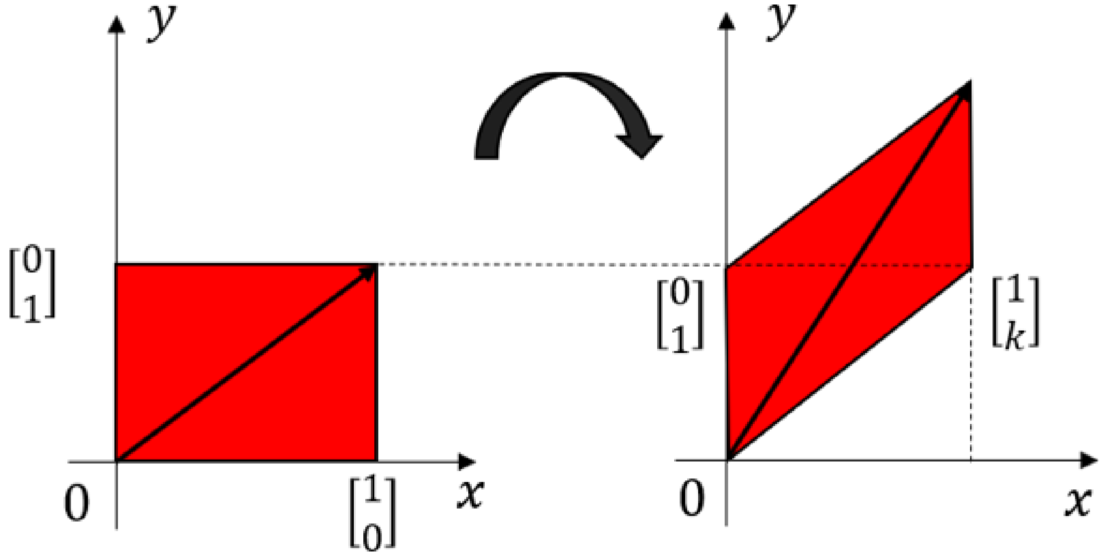
In the language of matrices, the horizontal shear will be given by:

$$k \quad x \quad \begin{matrix} 1 & h \\ 0 & 1 \end{matrix} \quad x$$

Vertical shear of h-factor :

$$C_k : R^2 \longrightarrow R^2$$

$$(x, y) \longrightarrow C_k\,(x, y) = (x, hx + y)$$



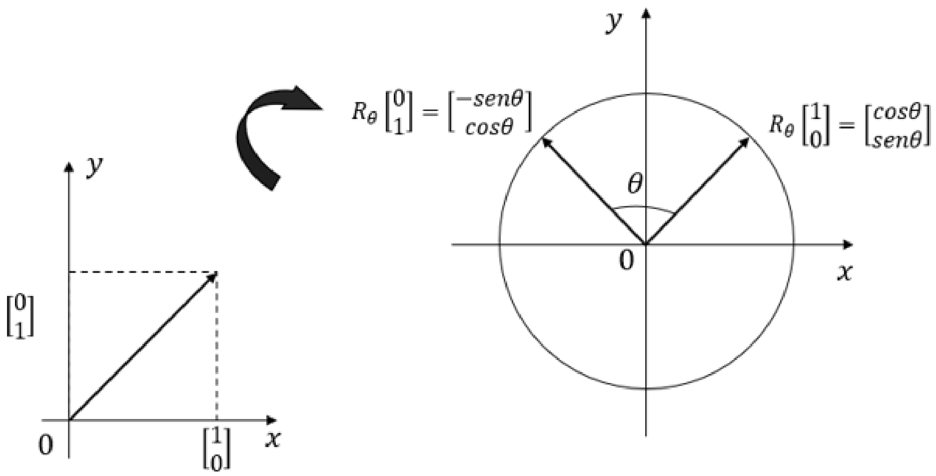
In the language of the matrices, the vertical shear of factor h will be delineated by:

$$k \quad x \quad \begin{matrix} 1 & 0 \end{matrix} \quad x$$

Anti-horory rotation of an angle θ :

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longrightarrow R_\theta (x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$



In the language of matrices, the counterclockwise rotation of an angle θ is given by:

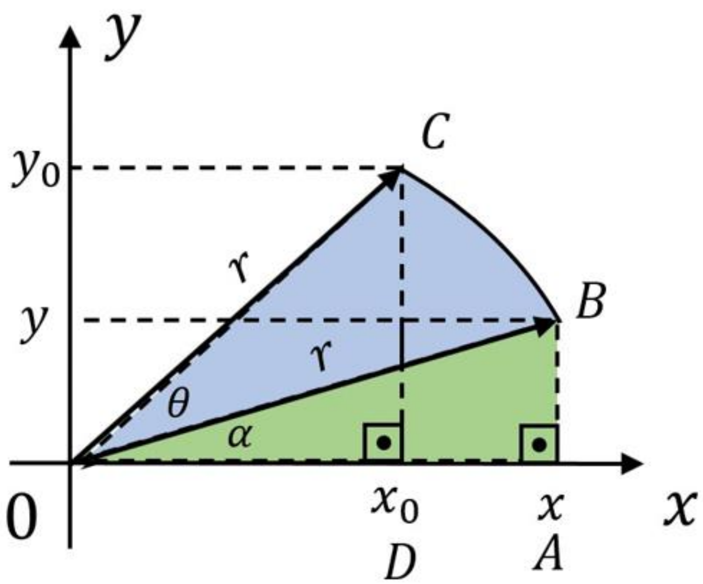
$$R_\theta \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Note:

Anti-horory rotation of an angle θ :

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longrightarrow R_\theta (x, y) = (x_0, y_0)$$



Please note that:

ΔOAB :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (1)$$

ΔOAB :

$$\begin{cases} x_0 = r\cos(\quad + \theta) \\ y_0 = r\sin(\quad + \theta) \end{cases} \qquad (2)$$

$$\cos(\quad + \theta) = \cos\quad \cos\theta - \sin\quad \sin\theta$$

Then, using the identities

$$\sin(\quad + \theta) = \sin\quad \cos\theta + \sin\theta\cos$$

$$x_0 = r\cos(\quad + \theta) = r\cos\quad \cos\theta - r\sin\quad \sin\theta \qquad (3)$$

e

$$y_0 = r\sin(\quad + \theta) = r\sin\quad \cos\theta + r\sin\theta\cos \qquad (4)$$

Now, substituting (1) into (3), (4), we get:

$$x_0 = r\cos(\quad + \theta) = x\cos\theta - y\sin\theta$$

e

$$y_0 = r\sin(\quad + \theta) = y\cos\theta + x\sin\theta.$$

So

$$R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Or rewriting the counterclockwise rotation in matrix form, we have:

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If the rotation is non-clockwise, simply replace θ with (—θ), highlighting that

$$\begin{cases} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{cases}$$

The hourly rotation matrix will be given by:

$$\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Theorem

Let you and B.vector spaces over R and let $T : U \rightarrow B$ be a linear transformation. T is injector if, and only if, Ker (T) = (0}

$$U \xrightarrow{T} B \text{ It's an injector} \iff Ker(T) = \{0\}$$

Demonstration:

Part 1: We want to show that:

$u_O \in$ Ker (T) and T is injector, so $u_O = 0$

(=⇒) In fact, $u_O \in Ker(T)$, so $T(u_O) = 0 = T(0)$.

Now, since T is injector, it follows that: $u_O = 0$.

Soon

$$Ker(T) = \{0\}.$$

Part 2:

Reciprocally, we want to prove that:

($\Longleftarrow$) Ker(T) = {0} and for all $u_1, u_2 \in U : T(u_1) = T(u_2) \Longrightarrow u_1 = u_2$, i.e. T is injector.

See

$$T(u_1) = T(u_2) \Longrightarrow T(u_1) - T(u_2) = 0$$
$$\Longrightarrow T(u_1 - u_2) = 0$$
$$\Longrightarrow (u_1 - u_2) \in Ker(T) = \{0\}$$
$$\Longrightarrow u_1 - u_2 = 0 \Longrightarrow u_1 = u_2.$$

So: T is an injector.

Note:

$$U \xrightarrow{T} B \text{ It's an injector} \Longleftrightarrow Ker(T) = \{0\}$$

This theorem is valid in infinite dimension, since at no time is dimension used in its proof.

Example1:

Prove that: C([0, 1]) is isomorphic to C([2, 3]), i.e., C([0, 1])' C[2, 3] .

It is worth noting that:

$$C([a,b]) = \{f : [a,b] \rightarrow R \quad \text{A continuous function}\}.$$

Proof:

In fact, $2 \leq x \leq 3 \Longleftrightarrow 0 \leq x - 2 \leq 1$, then we can take

$$\begin{aligned} T : \quad C([0,1]) &\rightarrow \qquad C[2,3] \\ f &\longrightarrow \quad T(f)(x) = f(x-2). \end{aligned}$$

It's easy to see that T is linear (Check!)

Let's prove that T is an isomorphism, for simplicity's sake let's do the following:

$$ę : \quad [2,3] \rightarrow \qquad [0,1]$$
$$x \longrightarrow \quad ę(x) = x - 2,$$

Similarly, the inverse of is given by:

$$ę^{-1} : \quad [0,1] \rightarrow \qquad [2,3]$$
$$(x-2) \longrightarrow \quad ę^{-1}(x-2) = x.$$

Also, look at the following diagram:

$$[0,1] \xrightarrow{ę^{-1}} [2,3]$$
$$\searrow \qquad M g$$
$$goę^{-1} \qquad R.$$

Statement 1: T is an injection molding machine

Let fO ∈ Ker (T), then T [fO] (x) = (fOoę) (x) = 0, hence let us dry that: fOoę. So

$$(f_o oę)\, oę^{-1} = 0.ę^{-1} = 0.$$

In other words, we have:

$$f_o o \;\; ęoę^{-1} \;\; = f_o = 0.$$

So: T is an injector.

Statement 2: T is superjective

$$6g \in C([2,3])\,,\ \mathbf{I}ę^{-1} \in C([0,1]) : goę^{-1} \in C([0,1]),$$ so that:

$$T \;\; goę^{-1} \;\; = go \;\; ę^{-1}oę \;\; = g.$$

Therefore, T is superjective.

Therefore, T is an isomorphism. In addition, we can write in another way:

$$C([0,1])\ '\ C[2,3].$$

Example 2:

Let $T: R^3 \rightarrow R^2$ be a linear transformation, defined by:

$$T(x, y, x) = (x - y, x + y).$$

The following are requested:

Check: T is injecting machine and get the dim Ker (T).

dim Fm(T) is a base for Fm(T). Is T a superjector?

Claim: T is not an injector.

In fact, the T nucleus is described by:

$$Ker(T) = \{(x_o, y_o, x_o) \in R^3; T(x_o, y_o, x_o) = (0, 0)\}.$$

$$T(x_o, y_o, x_o) = (x_o - y_o, x_o + y_o) = (0, 0)$$

$$\Longrightarrow \begin{cases} x_o - y_o = 0 \\ x_o + y_o = 0 \end{cases} \Longrightarrow x_o = y_o = 0.$$

So

$$Ker(T) = \{(0, 0, x_o) \in R^3; x_o \in R\} = [(0, 0, 1)].$$

It follows that: T is not an injection molding machine. Also, dim Ker (T) = 1.

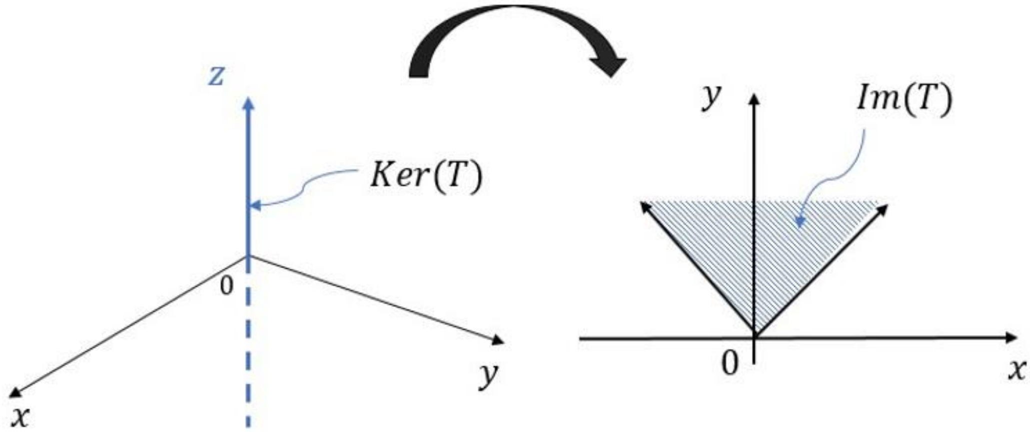
Now, by the nucleus and image theorem, we have:

$$3 = \dim R^3 = 1 + \dim Fm(T) \Longrightarrow \dim Fm(T) = 2$$

and Fm (T) $\subseteq R^2$. Thus, Fm(T) = R^2 and therefore T is superjective.

Let's determine the geradores for Fm (T) : T (x, y, x) = x (1, 1) + y (—1, 1) .

Hence the following: Fm (T) = [(1, 1) , (—1, 1)]. Since dim Fm(T) = 2, it follows that β = ((1, 1) , (—1, 1)} is a basis for Fm(T).



Eureka!!!
Core and image theorem

$$\dim \mathbb{R}^3 = 3$$
$$\dim Ker(T) = 1$$
$$\dim Im(T) = 2$$

$$dim\mathbb{R}^3 = \dim Ker(T) + dimIm(T)$$

Theorem : Let you and B be vector spaces over R, let U be of finite dimension [dim U < ∞] and let T : U →B be a linear transformation. We have then:

$$\dim U = \dim Ker(T) + \dim Fm(T)$$

Demo: See references [7, 10, 13, 16, 18]

Example:

Let T : D_1 (R) →R^2 be a linear transformation, defined by:

$$T(ax + b) = (b, a + b).$$

Prove that: T is an isomorphism, then get T^{-1} : R2→ D1(R).

Proof

Statement 1: $D_1(R) \xrightarrow{T} R^2$ is injector $\Longleftrightarrow$ Ker (T) = $\{0x + 0\}$.

A priori, by definition the nucleus of T is given by:

$$\mathrm{Ker}(T) = \{p_0(x) = a_0x + b_0 \in D_2(R); T(a_0x + b_0) = (0,0)\}.$$

See:

$$T(a_0x + b_0) = (b_0, a_0 + b_0) = (0,0) \Longrightarrow \begin{cases} b_0 = 0 \\ a_0 + b_0 = 0 \end{cases} \Longrightarrow \begin{cases} b_0 = 0 \\ a_0 = 0 \end{cases}$$

Therefore, Ker (T) = $\{p_0(x) = 0x + 0\} = \{0\}$ $\Longleftrightarrow$ T is injector.

Statement 2: T is superjective

In the light of the nucleus and image theorem we have:

$$2 = \dim D_1(R) = 0 + \dim \mathrm{Fm}(T)$$

and Fm (T) $\subseteq R^2$, whence comes: Fm (T) = R^2.

Therefore, T is superjective, and consequently we get: T is an isomorphism.

Now, let's find the inverse isomorphism:

$$T^{-1} : R^2 \to D_1(R).$$

$$T^{-1}(a_0, b_0) = h_1x + h_2 \Longleftrightarrow T(h_1x + h_2) = (a_0, b_0). \qquad (1)$$

In other words, we need to find h_1 and h_2 as a function of a_0 and b_0.

See

$$T(h_1x + h_2) = (h_2, h_1 + h_2) = (a_0, b_0) \Longrightarrow \begin{cases} h_2 = a_0 \\ h_1 + h_2 = b_0 \end{cases} \Longrightarrow \begin{cases} h_2 = a_0 \\ h_1 = b_0 - a_0. \end{cases} \qquad (2)$$

Thus, substituting (2) into (1) follows the inverse isomorphism:

$$T^{-1}(a_0, b_0) = (b_0 - a_0)x + a_0.$$

Comments

The compositions of linear transformations and their inverses produce the results

$$\begin{array}{ccc} D_1(R) & \xrightarrow{T} & R^2 \\ T^{-1} \circ T & \searrow & \downarrow T^{-1} \\ & & D_1(R). \end{array}$$

Hence, it comes:

$$T^{-1} \circ T \ (p(x)) = p(x).$$

Similarly, we have:

$$\begin{array}{ccc} R^2 & \xrightarrow{T^{-1}} & D_1(R) \\ T \circ T^{-1} & \searrow & \downarrow T \\ & & R^2 \end{array}$$

Therefore, we get:

$$T^{-1} \circ T \ (p(x)) = p(x).$$

Problem:

1. Find $A : R^2 \to R^2$ which is a linear transformation given by: a contraction of factor $\frac{1}{2}$ followed by a counterclockwise rotation of $\frac{\pi}{3}$ rad . Highlight $[A]$.

Solution:

$$\mathbb{R}^2 \xrightarrow{C\left(\frac{1}{2}\right)} \mathbb{R}^2$$

$$A = R_{(\pi/3)} \circ C_{\left(\frac{1}{2}\right)} \searrow \quad \downarrow R_{(\pi/3)}$$

$$\mathbb{R}^2.$$

Like this

$$[A] = \left[R_{(\pi/3)} \circ C_{\left(\frac{1}{2}\right)}\right] = \left[R_{(\pi/3)}\right]\left[C_{\left(\frac{1}{2}\right)}\right] =$$

$$= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix},$$

Thus, A in matrix form is given by:

$$[A]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x-\sqrt{3}y}{4} \\ \frac{\sqrt{3}x+y}{4} \end{pmatrix}.$$

Or even,

$$A(x,y) = \left(\frac{x-\sqrt{3}y}{4}, \frac{\sqrt{3}x+y}{4}\right).$$

2. Find $A : \mathbb{R}^2 \to \mathbb{R}^2$ which is a linear transformation given by: a counterclockwise rotation of $\frac{\pi}{4}$ rad followed by a factor expansion of $\sqrt{2}$. Highlight [A] .

Solution:

In fact, making the diagram for the composition, we have:

$$\mathbb{R}^2 \xrightarrow{R_{(\pi/4)}} \mathbb{R}^2$$

$$A = E_{(\sqrt{2})} \circ R_{(\pi/4)} \searrow \quad \downarrow E_{(\sqrt{2})}$$

$$\mathbb{R}^2.$$

Like this

$$[A] = \left[E_{(\sqrt{2})}\right]\cdot\left[R_{(\pi/4)}\right] = \left[E_{(\sqrt{2})}\right]\cdot\left[R_{(\pi/4)}\right] =$$

$$= \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Thus, A in matrix form is given by:

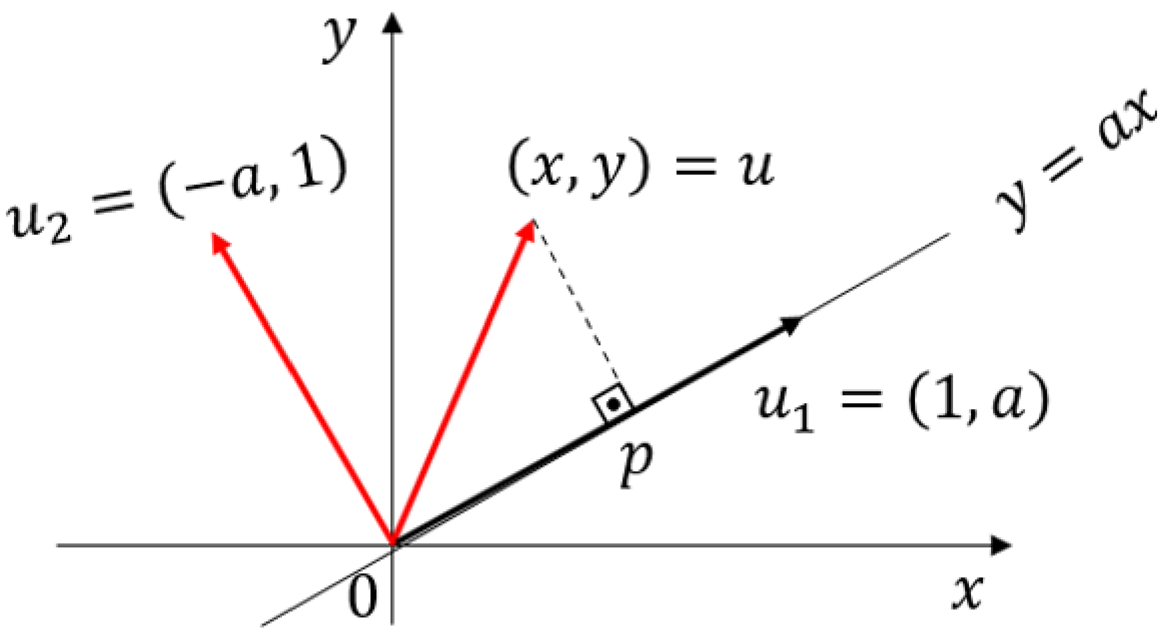
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$$

Or even,

$$A(x, y) = (x - y, x + y).$$

3. Find the orthogonal projection $P : \mathbb{R}^2 \to \mathbb{R}^2$ of a vector $u = (x, y)$ over the line 5: $y = ax$, $a \neq 0$ (Sketch the problem.)

Solution:



Note that $\beta = \{(1, a), (-a, 1)\}$ is a basis for $\mathbb{R}^2$, such that the orthogonal projection satisfies:

$$P(1,a) = (1,a) \text{ e } P(-a,1) = (0,0).$$

$\forall (x,y) \in R^2 : \exists \lambda_1, \lambda_2 \in R$, such as:

$$(x,y) = \lambda_1 (1,a) + \lambda_2 (-a,1). \qquad (1)$$

From this, we get:

$$\begin{cases} \lambda_1 - a\lambda_2 = x \\ a\lambda_1 + \lambda_2 = y \end{cases} \overset{aE_2 + E_1 \to E_2}{\Longrightarrow} \begin{cases} \lambda_1 - a\lambda_2 = x \\ (1+a^2)\lambda_1 = x + ay. \end{cases}$$

where, comes:

$$\begin{cases} \lambda_1 = \frac{x+ay}{1+a^2} \\ \lambda_2 = y - a\lambda_1 \end{cases} \Longrightarrow \begin{cases} \lambda_1 = \frac{x+ay}{1+a^2} \\ \lambda_2 = y - a\frac{x+ay}{1+a^2} \end{cases} \Longrightarrow \begin{cases} \lambda_1 = \frac{x+ay}{1+a^2} \\ \lambda_2 = \frac{y+a^2y-ax-a^2y}{1+a^2}. \end{cases} \qquad (2)$$

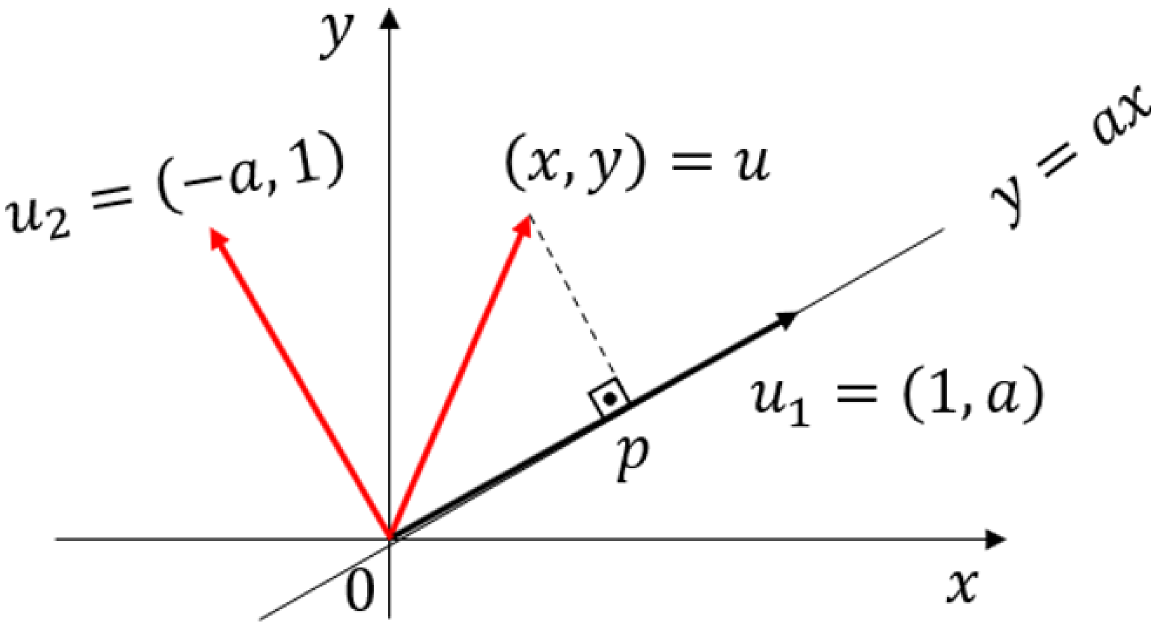
Now, by substituting (2) into (1), we get:

$$(x,y) = \frac{x+ay}{1+a^2}(1,a) + \frac{y-ax}{1+a^2}(-a,1).$$

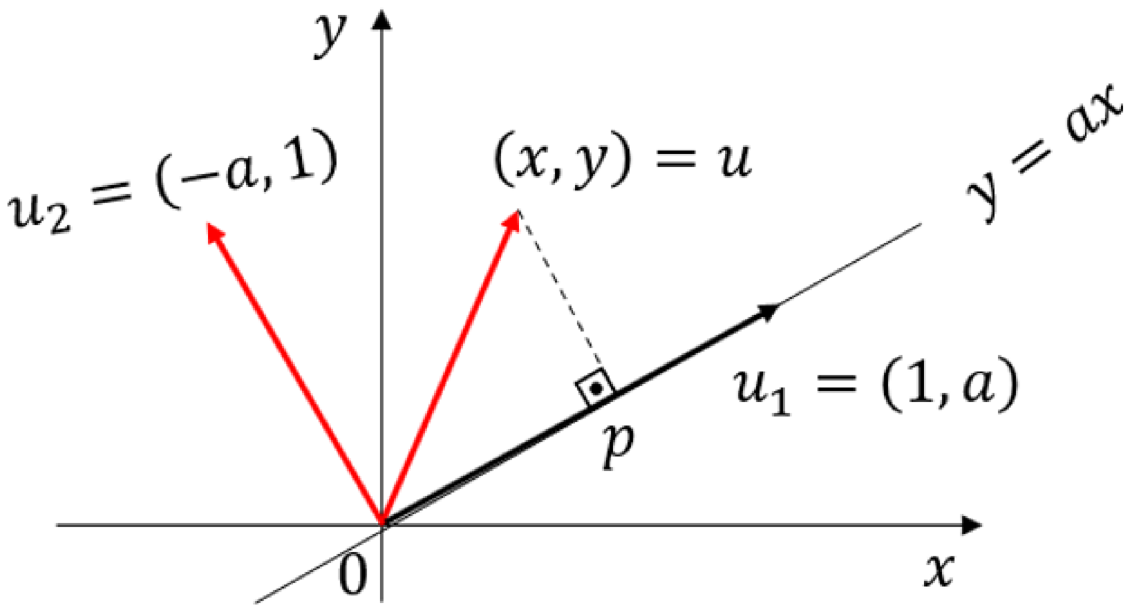
Applying P and its linearity, we get:

$$\begin{aligned} P(x,y) &= \frac{x+ay}{1+a^2} P(1,a) + \frac{y-ax}{1+a^2} P(-a,1) \\ &= \frac{x+ay}{1+a^2}(1,a) + \frac{y-ax}{1+a^2}(0,0) \\ &= \frac{x+ay}{1+a^2}, \frac{xa+a^2y}{1+a^2} . \end{aligned}$$

Note: Revisiting Vector Analytic Geometry:



If we use the orthogonal projection P of u = (x, y) over u_1 = (1, a), we immediately get :



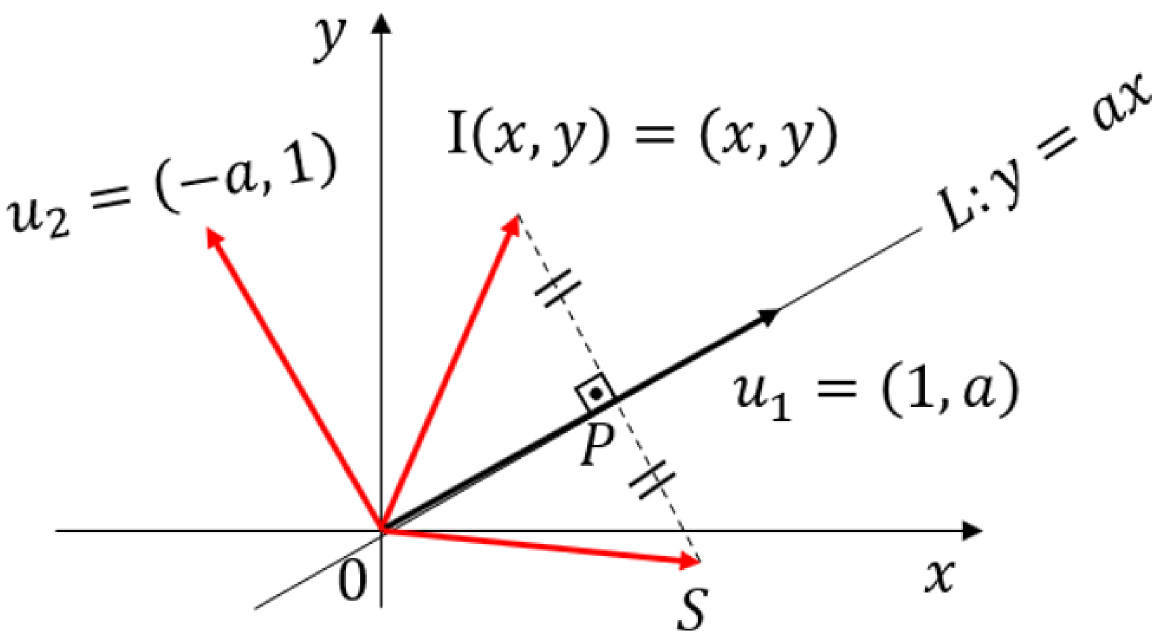
$$P^{u}_{u_1} = \frac{\langle u, u_1 \rangle}{\|u_1\|^2} . u_1 \Longrightarrow P^{u}_{u_1} = \frac{x + ay}{1 + a^2} . (1, a) = \frac{x + ay}{1 + a^2}, \frac{xa + a^2 y}{1 + a^2} \quad .$$

Rewriting the orthogonal projection in matrix form, we have:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{1+a^2} & \frac{a}{1+a^2} \\ \frac{a}{1+a^2} & \frac{a^2}{1+a^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

4. You will get reflection $S : R^2 \to R^2$ from a vector $u = (x, y)$ around the line $5 : y = ax$, $a \neq 0$.

Solution:



Suffice it to note that:

$$S - P = P + I \iff S(x, y) = 2.P(x, y) - I(x, y),$$

where the identity $I: R^2 \to R^2$ is given by: $I(x,y) = (x,y)$.

The matrix of reflection S is given by:

$$[S] = 2[P] - I_2 \iff [S] = 2.\begin{pmatrix} \frac{1}{1+a^2} & \frac{a}{1+a^2} \\ \frac{a}{1+a^2} & \frac{a^2}{1+a^2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Let $T : R^2 \to R^2$, be a linear transformation defined by:

$$T(x, y) = (x + y, y).$$

and let A = ((x, y) ∈ R^2; max (|x| , |y|} = 1} . Determine T (A).

Solution:

Auxiliary Calculations:

$$\begin{cases} T(1,0) = (1,0) \\ T(1,1) = (2,1) \\ T(0,1) = (1,1) \end{cases}, \quad \begin{cases} T(-1,0) = (-1,0) \\ T(-1,1) = (0,1) \\ T(-1,0) = (-1,0) \end{cases}, \quad \begin{cases} T(-1,0) = (-1,0) \\ T(-1,-1) = (-2,-1) \\ T(0,-1) = (-1,-1) \end{cases}$$

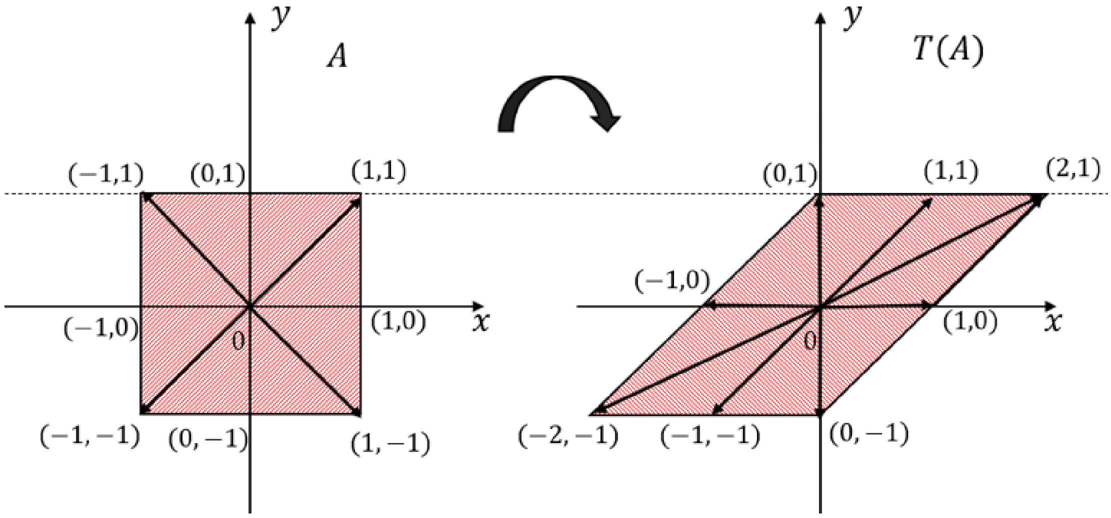
and

$$\begin{array}{c} T(1,0) = (1,0) \\ T(1,-1) = (0,-1) \\ T(-1,-1) = (-2,-1) \end{array}$$

In addition, we have

$$|x| = 1 \Longleftrightarrow \begin{cases} x = 1 \\ \text{ou} \\ x = -1 \end{cases} \quad \text{e} \quad |y| = 1 \Longleftrightarrow \begin{cases} y = 1 \\ \text{ou} \\ y = -1 \end{cases}$$

T transforms the square on side 1 into a paraelogram, geometrically, we have:



Let T : $R^2 \rightarrow R^2$, be a linear transformation, defined by:

$$T(x, y) = (x + y, x - y).$$

The following are requested:

(i) Prove by definition that T is an injector. (ii) Show that T is superjective.

Solution:

We want to show that: T : $R^2 \rightarrow R^2$ is an injection molding machine by definition:

$$6u_1, u_2 \in R^2 : T(u_1) = T(u_2) \Longrightarrow u_1 = u_2.$$

Consider that $u_1, u_2 \in R^2$, com $u_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$, then, tem-se:

$$T[u_1] = T[u_2] \Longrightarrow T(x_1, y_1) = T(x_2, y_2)$$

$$= (x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$$

$$\Longrightarrow \begin{cases} x_1 + y_1 = x_2 + y_2 \\ x_1 - y_1 = x_2 - y_2 \end{cases} \Longrightarrow \begin{cases} 2x_1 = 2x_2 \\ 2y_1 = 2y_2 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = x_2 \end{cases} \Longrightarrow u_1 = u_2.$$

So: T is an injector by definition.

We want to show that: T : $R^2 \rightarrow R^2$ is superjective by definition:

$$6v_0 = (a, b) \in R^2 : Iu_0 = (x_0, y_0) \in R^2 : T(u_0) = v_0$$

It is necessary to obtain x_O and y_O as a function of a and b.

See

$$T(u_0) = T(x_0, y_0) = (x_0 + y_0, x_0 - y_0) = (a, b).$$

Then you get it.

$$\begin{cases} x_0 + y_0 = a \\ x_0 - y_0 = b \end{cases} \implies \begin{cases} x_0 = \frac{a+b}{2} \\ y_0 = \frac{a-b}{2} \end{cases}$$

Like this

$$6v_0 = (a, b) \in R^2 : Iu_0 = \left(\frac{a+b}{2}, \frac{a-b}{2}\right) \in R^2 : T(u_0) = T(x_0, y_0)$$

$$= T\left(\frac{a+b}{2}, \frac{a-b}{2}\right) = \left(\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}\right) = (a, b) = v_0.$$

Therefore, it comes:

$$T(u_0) = v_0.$$

Or again, Fm (T) = R^2 and therefore T is superjective by definition.

Consequently, T is bijective.

5. Let $T : D_1(R) \rightarrow R^2$ be a transformation defined by:

$$T(p(x)) = (p(0), p(1)).$$

Prove that:

T is linear (b) T is injector (by definition). (s) Is T superjective? Justify!

Solution:

(i) 6p1, p2 ∈ D1 (R), we have:

$$T\,((p_1 + p_2)(x)) = ((p_1 + p_2)(0), (p_1 + p_2)(1))$$
$$= (p_1(0), p_1(1)) + (p_2(0), p_2(1))$$
$$= T\,(p_1(x)) + T\,(p_2(x))$$

6λ ∈ R, 6p1 ∈ D1 (R), we have:

$$T\,((\lambda p_1)(x)) = ((\lambda p_1)(0), (\lambda p_1)(1))$$
$$= \lambda(p_1(0), p_1(1))$$
$$= \lambda T\,[p_1(x)]$$

Therefore, T is linear.

We want to show that: T : D1(R) →R2 is an injection molding machine by definition:

$$6p_1, p_2 \in D_1(R) : T\,[p_1(x)] = T\,[p_2(x)] \Longrightarrow p_1(x) = p_2(x).$$

Let's consider p (x) = ax +b ∈ D1 (R) , we $\begin{cases} p(0) = b \\ p(1) = a + b \end{cases}$ T will be rewritten in the form:

$$T\,[p(x)] = T\,[ax + b] = (b, a + b).$$

$6p_1, p_2 \in D_1(R)$ com $p_1(x) = a_1x + b_1$ e $p_2(x) = a_2x + b_2$, has been:

$$T\,[p_1(x)] = T\,[p_2(x)] \Longrightarrow T\,[a_1x + b_1] = T\,[a_2x + b_2]$$
$$= (b_1, a_1 + b_1) = (b_2, a_2 + b_2) \Longrightarrow \begin{cases} b_1 = b_2 \\ a_1 + b_1 = a_2 + b_2 \end{cases}$$
$$\Longrightarrow \begin{cases} b_1 = b_2 \\ a_1 \quad a_2 \end{cases} \Longrightarrow p_1(x) = p_2(x).$$

So T is injector.

(s) T : Is D1(R) →R2 superjective? We want to show that:

$$6v_0 = (\ _0, \beta_0) \in R^2 : Ip_0(x) = a_0x + b_0 \in D_1(R) : T[p_0(x)] = v_0$$

That is, the image of T is equal to R^2 itself (There are no elements left in the range of T). We need to find a_O and b_O as a function of $_O$ and β_O.

Indeed

$$6v_0 = (\ _0, \beta_0) \in R^2 : Ip_0(x) = a_0x + b_0 \in D_1(R) :$$

$$T[p_0(x)] = T[a_0x + b_0] = (b_0, a_0 + b_0) = (\ _0, \beta_0)$$

$$\Longrightarrow \quad b_0 = \ _0 \quad \Longrightarrow \quad b_0 = \ _0$$

$$a_0 + b_0 = \beta_0 \qquad a_0 = \beta_0 - b_0$$

$$\Longrightarrow \quad b_0 = \ _0$$

$$a_0 = \beta_0 - \ _0 \ .$$

Like this

$$6v_0 = (\ _0, \beta_0) \in R^2 : I[p_0(x) = a_0x + b_0 = (\beta_0 - \ _0)x + \ _0] \in D_1(R) :$$

$$T[p_0(x)] = T[(\beta_0 - \ _0)x + \ _0] = (\ _0, (\beta_0 - \ _0) + \ _0) = (\ _0, \beta_0) = v_0.$$

Therefore, Fm (T) = R^2, i.e., T is superjective by definition.

Note:

$$T[p(x)] = T[ax + b] = (b, a + b).$$

Another, more far-fetched way of proving subjectivity will be given by:

$$T[p(x)] = T[ax + b] = (b, a + b) = b(1, 1) + (0, 1).$$

The image of T is generated by (1, 1) and (0, 1), hence we get:

$$Fm(T) = [(1, 1), (0, 1)].$$

Statement: r = ((1, 1) , (0, 1)} is linearly independent In fact, given the equation

$$\lambda_1 (1, 1) + \lambda_2 (0, 1) = (0, 0) \Longrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{cases} \Longrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0. \end{cases}$$

Therefore, r = ((1, 1) , (0, 1)} is linearly independent and therefore comes: r is a basis for R^2. In addition, dim Fm(T) = 2 (number of vectors of one of the bases) and Fm(T) ⊆ R2. Thus: Fm (T) = R^2, i.e., T is superjective.

Don't forget!

T : D1(R) →R2 is an injection molding machine and a superjector⇐⇒ T is an ejector

6. Let T : R^n→R, T ($x_1, x_2, \ldots, x_n$) $= \sum_{i=1}^{n} a_i x_i$: $a_1x_1 + \ldots +$ anxn a linear transformation. Prove that T is a superjector.

Solution

1ST MODE:

Indeed, for 6 ∈ R : IX = ($x_1, x_2, \ldots, x_n$) ∈ R^n, such that:

$$T(x_1, x_2, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n = \quad .$$

Therefore, Fm (T) = R, i.e., T is superjective.

2ND MODE:

Suffice it to note that:

$$T(x_1, x_2, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n = \sum_{i=1}^{n} a_i x_i.$$

Hence, it comes:

$$\dim Fm(T) = \begin{cases} 1 \\ 0 \quad \text{ou} \end{cases}$$

(It is not appropriate), because T would be identically null.

Therefore, dim Fm (T) = 1 and Fm (T) ⊆ R, from which we get: Fm (T) = R and therefore T is superjective.

Let T : U →B be a linear transformation. Prove that:

$$\text{(i) } T(0) = 0 \quad \text{(ii) } T(-v) = -T(v) \quad \text{(iii) } T(u-v) = T(u) - T(v)$$

$$\text{(iv) } T\left(\sum_{i=1}^{n} u_i\right) = \sum_{i=1}^{n} T(u_i)$$

Solution

Suffice it to note that:

$$T(0) = T(0) + 0 \text{ e } T(0) = T(0+0) = T(0) + T(0)$$

Soon

$$T(0) + T(0) = T(0) + 0 \Longrightarrow T(0) = 0.$$

Indeed,

$$0 = T(0) = T[v + (-v)] = T(v) + T(-v).$$

It follows that:

$$T(-v) = -T(v).$$

Note that:

$$T[u + (-v)] = T(u) + T(-v).$$

Now, from the previous item T (—v) = —T (v), therefore, we get:

$$T(u - v) = T(u) - T(v)$$

The proof will be by finite induction on n

For n = 2 :

$$T\left(\sum_{i=1}^{2} u_i\right) = T(u_1 + u_2) = T(u_1) + T(u_2) = \sum_{i=1}^{2} T(u_i).$$

Suppose valid for n (induction hypothesis), then it is missing to show for n + 1.

Indeed

$$T\left(\sum_{i=1}^{n+1} u_i\right) = T\left(\sum_{i=1}^{n} u_i + u_{n+1}\right) = T\left(\sum_{i=1}^{n} u_i\right) + T(u_{n+1})$$
$$= \sum_{i=1}^{n} T(u_i) + T(u_{n+1}) = \sum_{i=1}^{n+1} T(u_i).$$

So

$$T\left(\sum_{i=1}^{n+1} u_i\right) = \sum_{i=1}^{n+1} T(u_i).$$

Let $T : D_3(R) \to D_4(R)$ be a transformation. Defined by:

$$(Tp)(x) = xp(x + 1)$$

Prove that: T is linear.

Solution

$6 p_1, p_2 \in D_3(R)$, $6 \lambda_1, \lambda_2 \in R$, one has

$$T(\lambda_1 p_1 + \lambda_2 p_2)(x) = x(\lambda_1 p_1 + \lambda_2 p_2)(x+1)$$
$$= \lambda_1 x p_1 (x+1) + \lambda_2 x p_2 (x+1)$$
$$= \lambda_1 (T p_1)(x) + \lambda_2 (T p_2)(x).$$

So T is linear.

Let C ([a, b]) be the set of continuous functions $f : [a, b] \to R$. Define

$$T \qquad C([a,b]) \longrightarrow R$$
$$f \longrightarrow T[f(x)] = \int_a^b f(x)\,dx.$$

Prove that: T is linear.

Proof:

$6 f, g \in C([a,b])$, $6 \lambda_1, \lambda_2 \in R$, , we have

$$T(\lambda_1 f + \lambda_2 g)(x) = \int_a^b (\lambda_1 f + \lambda_2 g)(x)\,dx$$
$$= \lambda_1 \int_a^b f(x)\,dx + \lambda_2 \int_a^b g(x)\,dx$$
$$= \lambda_1 T[f(x)] + \lambda_2 T[g(x)].$$

So: T is linear.

Let $T : R^2 \to R^2$, be a T.L. such that: $T(1, 0) = (1, 1)$ and $T(1, 1) = (0, 3)$.

Determine:

T(x,y)

If T is an automorphism. If so, get $T^{-1}(x, y)$

Solution:

$6 (x, y) \in R^2 : I \lambda_1, \lambda_2 \in R : (x, y) = \lambda_1 (1, 0) + \lambda_2 (1, 1)$.

From there, it comes :

$$\begin{cases} \lambda_1 + \lambda_2 = x \\ \lambda_2 = y \end{cases} \Longrightarrow \begin{cases} \lambda_1 = x - y \\ \lambda_2 = y. \end{cases}$$

Like this

$$(x, y) = (x - y) \, . \, (1, 0) + y \, . \, (1, 1) \, .$$

Now, applying T and its linearity, we get:

$$\begin{aligned} T\,(x, y) &= T\,[(x - y) \, . \, (1, 0) + y \, . \, (1, 1)] \\ &= (x - y) \, .T\,((1, 0)) + y.T\,((1, 1)) \\ &= (x - y) \, .T\,(1, 0) + y.T\,(1, 1) \\ &= (x - y) \, . \, (1, 1) + y \, . \, (0, 3) \\ &= (x - y, x + 2y) \, . \end{aligned}$$

So

$$T\,(x, y) = (x - y, x + 2y) \, .$$

T is bijetora.se, and only if, T is injector and superjector.

Statement 1: $R^2 \xrightarrow{T} R^2$ it is an injector$\Longleftrightarrow$ Ker (T) = ((0, 0)}

By definition of T core, we have:

$$Ker\,(T) = \{(x_0, y_0) \in R^2 : T\,(x_0, y_0) = (0, 0)\}$$

Vejamos T (x_O, y_O) = (x_O — y_O, x_O+ 2y_O) = (0, 0). From there:

$$\begin{cases} x_0 - y_0 = 0 \\ x_0 + 2y_0 = 0 \end{cases} \xRightarrow{2E_1 + E_2 \rightarrow E_2} \begin{cases} x_0 - y_0 = 0 \\ 3x_0 = 0 \end{cases} \Longrightarrow \begin{cases} x_0 = y_0 \\ x_0 = 0. \end{cases}$$

Therefore:

Ker (T) = ((0, 0)} ⇐⇒ T is injector.

Also, dim Ker (T) = 0.

Now, in light of the nucleus and image theorem, we have:

$$2 = \dim R2 = 0 + \dim Fm\,(T) \Longrightarrow \dim Fm\,(T) = 2$$

and since Fm(T) ⊆ R^2 it follows that:

$$Fm\,(T) = R^2.$$

In other words, T is a superjector, and consequently we get: T is a bijector, or T is an automorphism. Now, let's find the inverse automorphism

Solution:

Get

$$T^{-1}(x, y) = (h_1, h_2) \Longleftrightarrow T\,(h_1, h_2) = (x, y)\,.$$

See

$$T\,(h_1, h_2) = (h_1 - h_2, h_1 + 2h_2) = (x, y)\,.$$

Hence it comes:

$$\begin{cases} h_1 - h_2 = x \\ h_1 + 2h_2 = y \end{cases} \xrightarrow{2E_1 + E_2 \to E_2} \begin{cases} h_1 - h_2 = x \\ 3h_1 = 2x + y \end{cases} \Longrightarrow \begin{cases} h_2 = h_1 - x \\ h_1 = \frac{2x+y}{3} \end{cases}$$

$$\Longrightarrow \begin{cases} h_2 = \frac{2x+y}{3} - x \\ h_1 = \frac{2x+y}{3} \end{cases} \Longrightarrow \begin{cases} h_2 = \frac{y-x}{3} \\ h_1 = \frac{2x+y}{3} \end{cases}.$$

Therefore, T^{-1} is an automorphism

$$T^{-1}(x, y) = \frac{2x + y}{3}, \frac{y - x}{3} .$$

Appendix

Theorem:

The space of the linear transformations of U into B, such that: dim U =n and dim B =m is isomorphic to the space of matrices of order m × n with real inputs, i.e., 5 (U; B) is isomorphic to Mmvn (R) and is denoted by:

$$5(U; B) \text{ ' } M_{mvn}(R)$$

The fixation of the bases β c U and β'c B therefore determines a transformation of the

$$\Phi : \quad 5(U; B) \longrightarrow M_{mvn}(R)$$
$$f \longrightarrow \quad \Phi(f) = [A]^{\beta}_{\beta J} = B(f)$$

DEMONSTRATION

Statement 1: Φ is linear

$$\text{(i) } \Phi(f_1 + f_2) = [A_1 + A_2]^{\beta}_{\beta J} = [A_1]^{\beta}_{\beta J} + [A_2]^{\beta}_{\beta J} = \Phi(f_1) + \Phi(f_2)$$
$$\text{(ii) } \Phi(\lambda f_1) = [\lambda A_1]^{\beta}_{\beta J} = \lambda [A_1]^{\beta}_{\beta J} = \lambda\Phi(f_1) .$$

Therefore, Φ is linear.

Claim 2: Φ is an injection molding machine

In fact, Ker (Φ) = (f ∈ 5 (U; B) : Φ (f) = 0} .

Let's look at Φ $= [A]^{\beta}_{\beta J} = B(f) = 0,$

As $[f(u)]_{\beta J} = [A]^{\beta}_{\beta J} [u]_{\beta} = 0$, it follows that:

f (u) = 0.6u ∈ U. Consequently comes: f Ξ 0.

Therefore, Ker (Φ) = (0}, or again, Φ is injector.

Statement 3: Φ is superjective

6B ∈ M_{mvn} (R), If ∈ 5 (U; B) , such that:

$$\Phi(f) = [A]^{\beta}_{\beta J} = B(f) = B$$

Let us consider $f : U \longrightarrow B$, such that:

$$f(u_1) = b_{11}v_1 + b_{21}v_2 + \ldots + b_{m1}v_m$$
$$f(u_2) = b_{12}v_1 + b_{22}v_2 + \ldots + b_{m2}v_m$$

———— — — — — — — — — — — — —

$$f(u_n) = b_{1n}v_1 + b_{2n}v_2 + \ldots + b_{mn}v_m.$$

So we have:

$$B \quad \ldots \quad n \quad = B. \quad {}_{mvn}$$

So: Φ is a superjector. Therefore, we get: 5 (U; B) is isomorphic to $M_{mvn}(R)$.

Corollary

Let you and B be two vector spaces over R such that:

dim U = n and dim B = m. So, the space 5 (U; B) has dimension m.n

Demonstration

Let β and β' be the bases of you and B respectively.

A ——→ B, Φ: 5 (A; B) ——→ Mmxn (R), then we have, $\Phi(T) = [T]^{\beta}_{\beta J}$ $\Longrightarrow$ dim 5 (A; A; B) = dim Mmxn (R) = m.n

CONCLUDED

This mathematical look of a creative and flexible increasing mentality can be given as geometrical interpretations of the linear transformations of the plane into the plane in the language of the matrices of order 2, without any increase in difficulty in mathematical literacy . It is worth mentioning that: the compositions of transformations of the plane in the plane, serves as a first model of computer graphics.

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