SEVEN publicações acadêmicas

Variational or Variance Calculus

Crossref doi
https://doi.org/10.56238/sevened2023.006-155

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ABSTRACT
This research material suggests the exploration of approaches to deal with variational problems through approximation techniques. In mathematical contexts, variational problems involve optimization of functions, and approximation methods seek to find approximate solutions to these problems. These

approaches can be essential in situations where finding an exact solution is challenging or impractical, allowing for the effective analysis and resolution of complex issues through approximation techniques.

Keywords: Approximation methods, infinite series, MATLAB, boundary conditions.

1 INTRODUCTION

Previously (see volume 1 of this work) functional was defined as any numerical function established on a linear space $\mathcal{L}$. It could also be said that functional are the variable magnitudes whose values are determined by the choice or choice of one or more functions.

This chapter has been added to this volume to give a stronger foundation when presenting variational and energetic methods.

Before beginning to describe the *Calculus of Variations*, we will present some concepts and definitions that are believed to be useful to better develop the understanding of the object of this chapter.

Definition 1: Let $x \to x_0$ *denote x approaches x0 from the left and let be* $x \to x_0^+$ *denote x approaches* x0 from the right If

$$\lim_{x \to x_0} f\ x \neq \lim_{x \to x_0^+} f\ x \tag{1.1.1}$$

It is said that $f\,x$ is discontinuous in x_0, otherwise it is continuous in x_0.

Definition 2: A function is said to be continuous by parts in any interval if it has a finite number of discontinuities in the given interval.

Definition 3: A function is said to be differentiable at *x0* if the limit exists. It is said to be part differentiable in any interval if it has a derivative on the right and left for every inner point of the interval under consideration, and, moreover, these derivatives are equal except at a finite number of points.

$$\lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} \right\} \tag{1.1.2}$$

Definition 4: Let be the function $v = f\ \ x_1, x_2, \ldots, x_n$ where each variable is a function of other variables, like this $x_i = xi\ \ u_1, u_2, \ldots, u_n$. Then the partial derivative of *v* with respect to a given variable *ui* is given by:

$$\frac{\partial v}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j}{\partial u_i} \tag{1.1.3}$$

Resolution 5: The quantity

$$p\ x, y\ \ + q\ \ x, y\ \frac{dy}{dx} \tag{1.1.4}$$

is the derivative *dg/dx of some* function g xy *of some* function $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$. At this event, that is

$$p = \frac{\partial g}{\partial x} \text{ e } q = \frac{\partial g}{\partial y}$$

$$\frac{dg}{dx} = \frac{\partial g(x, y)}{\partial x} + \frac{\partial g(x, y)}{\partial y} \cdot \frac{dy}{dx} \tag{1.1.5}$$

Resolution 6: If

$$I = I(\varepsilon) = \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} f(x, \varepsilon) dx \tag{1.1.6}$$

So

$$\frac{dI}{\partial \varepsilon} = I\ \ '(\varepsilon) = f(x_2, \varepsilon)\frac{dx_2}{d\varepsilon} - f(x_1, \varepsilon)\frac{dx_1}{d\varepsilon} + \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} \frac{\partial f}{\partial \varepsilon} dx \tag{1.1.7}$$

If, and only if,

ensures that $\partial f / \partial \varepsilon$ is a continuous function of € and *x* in $[x_1, x_2]$. In case *x1* and *x2* are strictly constant, i.e., independent of €, the right-hand side of the above expression reduces to its final term since $\partial x_1 / \partial \varepsilon \equiv 0; \quad \partial x_2 / \partial \varepsilon \equiv 0$

Definition 7: In order to repeatedly employ the piecewise integration rule, it is necessary and sufficient that the functions *f* and *g* are only partially differentiable in the given interval

$$\int_{x_1}^{x_2} g \frac{df}{dx} dx = g.f \,|_{x_1}^{x_2} - \int_{x_1}^{x_2} f \frac{dg}{dx} dx \tag{1.1.8}$$

Definition 8: A function $F\ x_1,x_2,...,x_m,x_{m+1},...,x_{m+n}$ s said to be homogeneous of degree n, in the variables $x_{m+1},...,x_{m+n}$ if, for an arbitrary constant h , we have:

$$F\ x_1,x_2,...,x_m,hx_{m+1},...,hx_{m+n} = h^n F\ x_1,x_2,...,x_m,x_{m+1},...,x_{m+n} \quad (1.1.9)$$

Any function for which the above expression is valid satisfies Euler's theorem:

$$x_1 \frac{\partial F}{\partial x_1} + ... + x_{m+n} \frac{\partial F}{\partial x_{m+n}} = nF(x_1,x_2,...,x_m,...,x_{m+n}) \quad (1.1.10)$$

Definition 9: The required condition for a minimum (or maximum) of a function $F\ x_1,x_2,...,x_n = 0$ in relation to the variables, $x_i = x_1,x_2,...,x_n$ it satisfies the relationships:

$$G\ x_1,x_2,...,x_n = C_k, \forall k = 1,2,..,N \quad (1.1.11)$$

and

$$\frac{\partial F}{\partial x_i} = 0, \forall i = 1,2,..,n \quad (1.1.12)$$

where Ck are constants ar$F^+ = F + \sum_{k=1}^{n} \lambda_k G_k$:ed as unknow $\lambda_1,...,\lambda_n$ .lled Lagrange multipliers, are calculated together with the minimization (or maximization) values of by means of a set of equations formed by (1.1.11) and (1.1.12). $x_i, \forall i = 1,2,..,n$

Resolution 10: The line integral of a function f(*x, y, z*) from point P1 to point P2 along a curve C is defined by:

$$I = \lim_{n\to\infty} S_n = \int_C f(x,y,z)ds \quad (1.1.13)$$

Where $S_n = \sum F(x_k,y_k,z_k)\ \Delta S_k$ being ΔS_k length of the arcs of the C curve between the points

$x_{k-1},y_{k=1},z_{k-1}$ and x_k,y_k,z_k The integral (1.1.13) can also be represented in the form:

$$\begin{aligned} I_x &= \int_C f(xx,y,zz)dx \\ I_y &= \int_C f(x,y,z)dy \\ I_z &= \int_C f(x,y,z)dz \end{aligned} \quad (1.1.14)$$

$x = x\ t\ ,y = y\ t\ ,z = z\ t$

Introducing parametric equations where t grows

In the direction of growth of s , one can calculate (3) by the definite integral

$$I = \int_{t_1}^{t_2} f[x(t),y(t),z(t)]\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt \qquad (1.1.15)$$

With. $t_1 < t_2$

An important example of a line integral is made counterclockwise over a sharp curve in the xy plane. In this case, the parameter t is chosen in such a way that at the $[x\ t\ ,y\ t\ ]$ run the curve C counterclockwise when t grows from t1 to t2 , The integral above is equal to the area contained in C.

$$I = \frac{1}{2}\int_C (xdy - ydx) = \frac{1}{2}\int_{t_1}^{t_2} (x\frac{dy}{dt} - y\frac{dx}{dt})dt \qquad (1.1.16)$$

Resolution 11: Changing variables $x = x\ \ u,v,w\ \ y = y\ \ u,v,w$ and $z = z\ \ u,v,w$

In the calculation of the triple integral is done by:

$$\iiint_{\Omega} F(x,y,z)dxdydz = \iiint_{\Omega^*} f(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \qquad (1.1.17)$$

where f *is the function F expressed in terms of* u,v,w where Ω* is the *region Ω described by the variables* u,v,w *and where* $\left|\partial\ \ x,y,z\ \ /\partial\ \ u,v,w\ \ \right|$ *is the Jacobian.*

Resolution 12: If $z = z\ \ x,y$ is a continuously differentiable function of x and y , the area of a portion of the surface represented by this function is given by

$$I = \iint_{\Omega}\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dxdy \qquad (1.1.18)$$

where the integration is performed over a domain Ω the x-y plane over which a portion of the projected surface is given.

Be Ω a closed and limited region in the plan xy whose outline con$\partial\Omega$s of finite very smooth curves. Are $f\ x,y$ and $g\ x,y$ continuous functions having partial derivatives with respect to x and y in some subdomain contained in Ω . Then the integral exists along the entire Ω contour, such

$\partial\Omega$

that it is on the left when moving in the direction of integration, i.e., the line integral must be evaluated counterclockwise.

$$\iint_{\Omega}\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy=\oint_{\partial\Omega} fdx+gdy \tag{1.1.19}$$

Example

Let *w x,y be a* continuous function with continuous second-order partial derivatives in a domain Ω of the xy plane , of the type indicated by theorem 1.

$$f=-\frac{\partial w}{\partial y};g=\frac{\partial w}{\partial x} \tag{1.1.20}$$

Let them be $\partial f/\partial y$ e $\partial g/\partial dx$ then, exist and are contained in Ω. Let be the Laplacian of *w*:

$$\nabla^2 w=\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2} \tag{1.1.21}$$

The right-hand side of the integral equation that defines Green's theorem in the plane can be developed by the definition of the line integral, like this

$$\oint_{\partial\Omega}(fdx+gdy)=\int_{\partial\Omega}\left(f\frac{dx}{ds}+g\frac{dy}{ds}\right)ds=\int_{\partial\Omega}\left(-\frac{\partial w}{\partial y}\frac{dx}{ds}+\frac{\partial w}{\partial x}\frac{dy}{ds}\right)ds \tag{1.1.22}$$

where *s* is the arc length of $\partial\Omega$. The integrator of the last integral above, can be written as the scalar product of two vectors:

$$\nabla w=grad\ \ w=\frac{\partial w}{\partial x}\mathbf{i}+\frac{\partial w}{\partial y}\mathbf{i} \tag{1.1.23}$$

and

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} + \frac{dx}{ds}\mathbf{i} \tag{1.1.24}$$

that is

$$\nabla w \cdot \mathbf{n} = -\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds} \tag{1.1.25}$$

where ***n*** is a normal vector $\partial\Omega$ since the tangent vector a $\partial\Omega$

$$\mathbf{u} = \left(\frac{d\mathbf{r}}{ds}\right) = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \tag{1.1.26}$$

is orthogonal to ***n***that is ***n*** . ***u*** $= 0$. On the other hand, the product scalar $\nabla w \cdot \mathbf{n} = grad \quad w$. ***n*** is the directional derivative of w in the direction of n; denoting this scalar product by $\partial w / \partial \mathbf{n}$ and taking the expressions found in the formula of Gauss's theorem, we have

$$\iint_{\Omega} \nabla^2 w d\Omega = \int_{\partial\Omega} \frac{\partial w}{\partial n} ds = \int_{\partial\Omega} \nabla \mathbf{w} \bullet \mathbf{n} \;\; ds \tag{1.1.27}$$

Also, on the expression of Green's theorem, if and, $g = \eta P \;\; f = -\eta Q$ we get:

$$\iint_{\Omega} \left(P\frac{\partial\eta}{\partial x} + Q\frac{\partial\eta}{\partial y}\right) d\Omega = -\iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\eta d\Omega + \int_{\partial\Omega} Pdy - Qdx \tag{1.1.28}$$

which is the integration formula for double integrals. If in the above expression you do $\eta = \psi; P = \partial\phi / \partial x ; Q = \partial\phi / \partial y$ and from analogous modulus as above, we find an important result of Green's theorem:

$$\iint_{\Omega} \psi \nabla^2 \phi d\Omega = -\iint_{\Omega} \left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}\right) d\Omega + \int_{\partial\Omega} \psi \frac{\partial\phi}{\partial\eta} ds \tag{1.1.29}$$

1.1 FUNDAMENTAL THEOREM OF CALCULUS

Two forms of the fundamental theorem of calculus will be used, the first for the function-gradient pair and the other for the gradient-Hessian pair.

Motto 1a: Be f Twice continuously differentiable $f \in \mathbb{C}^2\ x$ in the vicinity of a line segment between the points and ξ So if you have $\mathbf{x} = \xi + \varepsilon, \in \mathbb{R}^n$

$$f\ \mathbf{x} = f\ \xi + \int_0^1 \nabla f\ \xi + t\mathbf{e}\ ^T \mathbf{e}dt; \qquad (1.1.30)$$
$$\nabla f\ \mathbf{x} = \mathbf{g}\ \mathbf{x} = \nabla f\ \xi + \int_0^1 \nabla^2 f\ \xi + t\mathbf{e}\ \mathbf{e}dt$$

Lema 1b: Let f be twice continuously differentiable $f \in \mathbb{C}^2\ x$ in the vicinity of a point $\xi \in \mathbb{R}^n$ so for $\mathbf{h} \in \mathbb{R}^n$ with $\|\mathbf{h}\| \ll \mathbf{1}$ small enough, if you have

$$f\ \mathbf{x}^* + \mathbf{h} = f\ \xi + \nabla f\ \xi\ \mathbf{h} + \frac{1}{2}\mathbf{h}^T \nabla^2 f\ \xi + o\ \|\mathbf{h}\|^2 \qquad (1.1.31)$$
$$= f\ \xi + \mathbf{g}\ \xi\ \mathbf{h} + \frac{1}{2}\mathbf{h}^T H\ \xi + o\ \|\mathbf{h}\|^2$$

At this point it would be good for the reader to read chapter 2 of volume 1 of this work (reference [4]), specifically in sections 2.6 to 2.8. By way of recall, some definitions already established in the above-mentioned reference are presented.

1.2 FUNCTIONAL

Resolution 13: Let be a set, $S \subseteq \mathbb{R}^n$ a subset, $X \subseteq S$, e $\mathbb{R}$ and the set of the real numbers. Defines itself as functional f about X to mapping f :X $\rightarrow \mathbb{R}$ **ctional logo is a numeric function** whose set of values of f is given by Rf:

$$R_f = k \in \mathbb{R} \mid f\ \mathbf{x} = k \therefore \mathbf{x} \in X \qquad (1.1.32)$$

In this chapter we will always have $S = \mathbb{R}^n$ the vector space of n dimensions. Thus, a typical of element $S \subseteq \mathbb{R}^n$ is written as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \therefore x_i \in \mathbb{R}$$

ε

Definition 14: For neighborhood of **x** with radius ε as a subset $V \subseteq S$, given by:

$$V(\mathbf{x},\varepsilon) = \{y \in \mathbb{R}^n \mid 0 \leq \|\mathbf{x}-\mathbf{y}\| < \varepsilon\} \tag{1.1.33}$$

Where ‖.‖ denotes the norm of Euclidean space defined for all $\mathbf{x} \in \mathbb{R}^n$ by:

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{2}}$$

Definition 15: Let a functional *f* be defined on $X \subset \mathbb{R}^n$ So $\xi \in X$ is a local minimizer of *f* if there is $\varepsilon > 0$ such a that

$$f(\xi) \leq f(\mathbf{x}), \forall \mathbf{x} \in V(\xi,\varepsilon) \cap X \tag{1.1.34}$$

If $f(\xi) < f(\mathbf{x}), \forall \mathbf{x} \in V(\xi,\varepsilon) \cap X, \mathbf{x} \neq \xi$ then ξ is a local strong minimizer of f Similarly, one can define local maximizer and local strong maximizer.

Resolution 16: (global minimizer). Be a functional *f* set on $X \subset \mathbb{R}^n$. So $\xi \in X$ It's a Global Enemy from *f* if there is a $\varepsilon > 0$ such that

$$f(\xi) \leq f(\mathbf{x}), \forall \mathbf{x} \in X \tag{1.1.35}$$

1.3 GRADIENT

Resolution 17:(Gradient). Be $f \in C^1(X)$ then the gradient of *f* in $\mathbf{x} \in X$ is the vector given by

$$g(\mathbf{x})^T = \nabla f^T = \left[\partial f/\partial x_1 \quad \partial f/\partial x_2 \quad \cdots \quad \partial f/\partial x_n\right] \tag{1.1.36}$$

If $f(\xi) < f(\mathbf{x}), \forall \mathbf{x} \in X, \mathbf{x} \neq \xi$ then ξ is a strong global minimizer of f.
Similarly, one can define global maximizer and global strong maximizer.

Analyzing the behavior of the quadratic functional in the open neighborhood (open *bal* ≤ mı *local minimizer* ξ using Taylor's ξ expansion over as a basic tool, it will be seen that two entities appear depending on whether the given functional belongs to $C^1(X)$ or $C^2(X)$ – where $C^m(X)$ represents the set of functionals for which partial order derivatives exist and are continuous in $X \subset \mathbb{R}^n$

The first entity is called a gradient (defined by $\nabla f(\mathbf{x}) \in C^1(X)$) and the second is called Hessian $H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) \in C^2(X)$

1.4 HESSIAN

Resolution 18:(Hessian) Be $f \in C^2(X)$ So the Hessian of *f* in $\mathbf{x} \in X$ is the symmetric matrix given by

$$H(\mathbf{x}) = \nabla^2 f = [h_{ij}]_{i,j=1}^{n}, \quad \therefore h_{ij} = \partial^2 f / \partial x_i \partial x_j \tag{1.1.37}$$

Or

$$H(\mathbf{x}) = \nabla^2 f = \begin{bmatrix} \partial^2 f/\partial x_1 \partial x_1 & \partial^2 f/\partial x_1 \partial x_2 & \cdots & \partial^2 f/\partial x_1 \partial x_n \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2 \partial x_2 & \cdots & \partial^2 f/\partial x_2 \partial x_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial^2 f/\partial x_n \partial x_1 & \partial^2 f/\partial x_n \partial x_2 & \cdots & \partial^2 f/\partial x_n \partial x_n \end{bmatrix}$$

Taylor's expansion of *the functional f* over **x** is conveniently expressed:

If there will be $f \in C^1$

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + g^T(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|) \tag{1.1.38}$$

If X, by $f \in C^2$

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + g^T(\mathbf{x})\mathbf{h} + \tfrac{1}{2}\mathbf{h}^T H(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|^2) \tag{1.1.39}$$

Note that in general, if f $\in C^m(X)$,,you will have

$$\begin{aligned} f(\mathbf{x}+\tau\mathbf{y}) &= \sum_{k=0}^{m-1} \frac{\tau^k}{k!} f^k(\mathbf{x};\mathbf{y}) + \frac{\tau^m}{m!} f^m(\mathbf{x}+\theta\tau\mathbf{y};\mathbf{y}) \\ \therefore f^m(\mathbf{x};\mathbf{y}) &= \left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + \cdots + y_n\frac{\partial}{\partial x_n}\right)^m f(\mathbf{x}) \end{aligned} \tag{1.1.40}$$

Resolution 19: Be $f \in C^1(X)$ It's said to be stationary in $\xi \in X$ if we have the gradient at this zero point, i.e., $\mathbf{g}(\xi) = 0$. This is equivalent to saying that if at any point on $\xi \in X$ functional the $f \in C^1(X)$ gradient (first derivative) is zero, then at $\xi \in X$ one has a stationary point.

Theorem 2: Let $f \in C^1$ X $\xi \in X$ If is a local minimizer of f , then f is stationary at $\xi \in X$

Proof:

In, $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + g^T(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|)$ putting $x = \xi$, $h = -\eta g(\xi)$ $\therefore h \in \mathbb{R}$

then $f(\mathbf{x}+\mathbf{h}) = f(\xi) - \eta\|g(\xi) \to g(\xi) \neq 0$ Assuming f is not stationary in ,

then for *sufficiently small values of n* one has to , $-\eta\|g(\xi)\|^2 + o(\|\mathbf{h}\|) < 0$ i.e. , $f(\mathbf{x}+\mathbf{h}) < f(\xi)$ in which case it cannot be a local minimizer .

Theorem 2 above is a necessary but not sufficient condition for it to be ξ a local minimizer. The sufficient condition is given by the following theorem:

Theorem 3: Let $f \in C^1$ X and let f be stationary in $\xi \in X$.

Then $\xi \in X$ it is a strong f local minimizer if the Hessian H ξ matrix is positive-defined (SPD)

Remember that it was learned in Linear Algebra that the necessary and sufficient conditions for a symmetric matrix A to be positive defined are:

1 $\mathbf{x^T A x} > 0, \forall \mathbf{x} \neq \mathbf{0}$

2 . $\forall \lambda_i = eig(A) \to \lambda_i > 0$

3 All upper submatrices Ak of A must have det $A_k > 0$

4 All di´s pivot (without line swapping) has to be $d_i > 0$

Proof:

Fror $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + g^T(\mathbf{x})\mathbf{h} + \frac{1}{2}\mathbf{h}^T H(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|^2)$ assuming the gradient is null

$\mathbf{g}(\xi) = 0$ si tem

$$
\begin{aligned}
f(\xi+\mathbf{h}) &= f(\xi) + \cancel{g^T(\xi)}\mathbf{h} + \frac{1}{2}\mathbf{h}^T H(\xi)\mathbf{h} + o\left(\|\mathbf{h}\|^2\right) \\
&= f(\xi) + \frac{1}{2}\mathbf{h}^T H(\xi)\mathbf{h} + o\left(\|\mathbf{h}\|^2\right)
\end{aligned}
\quad (1.1.41)
$$

Since by hypothesis H ξ is definite positive, then there exists a positive number λ_i (which can be given by the smallest eigenvalue of H ξ such that

$$\mathbf{h}^T H \xi \mathbf{h} \geq \lambda_i \|\mathbf{h}\|^2, \quad \forall \mathbf{h} \in \mathbb{R}^n .$$

Like this $\xi + \mathbf{h} = f \ \xi + \frac{1}{2}\lambda_i \|\mathbf{h}\|^2 + o \ \|\mathbf{h}\|^2$ and its right-hand part must be positive for sufficiently small values of $\|h\|$, from which one takes $f \ \xi + \mathbf{h} > f \ \xi$ in some vicinity of ξ.

Another form of the previous Theorem 3 is:

Theorem 3a: Let $f \in C^1 \ X$ and be *stationary* in $\xi \in X$.

So $\xi \in X$ It's a Local Strong Minimizer from *f* if or gradient g $\xi = 0$ and the matrix of the Hessian H ξ is positive-defined. Be then $h \in \mathbb{R}^n$ datum; by Taylor's theorem, for small values of $t \in \mathbb{R}$ such that $f \ \xi + th - f \ \xi \geq 0$ soon:

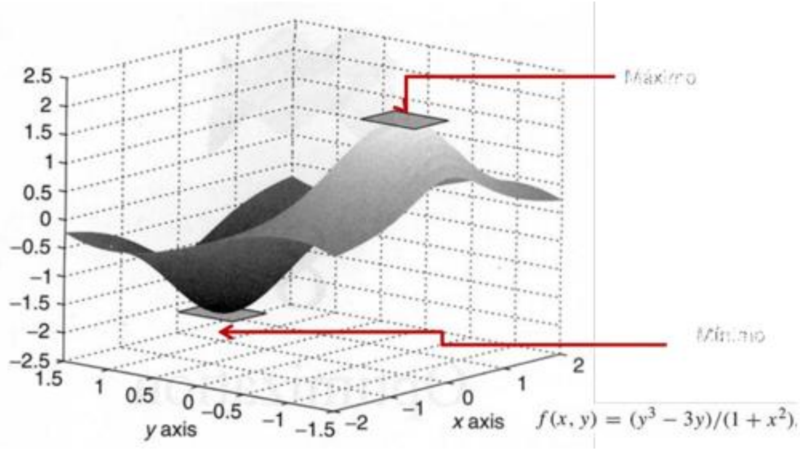
$$\begin{aligned}
&f \ \xi + th = f \ \xi + t\mathbf{g}^T \ \xi \ h + \tfrac{1}{2} t^2 h^T H \ \xi \ h + o \ t^2 \\
&\therefore \begin{cases} \mathbf{g}^T \ \xi = \nabla f \ \xi^{\ T} \\ H \ \xi = \nabla^2 f \ \xi \end{cases} \\
&\rightarrow \mathbf{g}^T \ \xi \ \mathbf{h} + \tfrac{1}{2} t^2 \mathbf{h}^T H \ \xi \ \mathbf{h} + o \ t \geq 0 \\
&\text{Se } t = 0 \text{ e } \mathbf{h} = -\mathbf{g} \ \xi \text{ se obtém: } \|\mathbf{g} \ \xi\|^2 = 0 \Rightarrow \mathbf{g} \ \xi \equiv 0, \text{ logo} \\
&\tfrac{1}{2} t^2 \mathbf{h}^T H \ \xi \ \mathbf{h} \geq 0 \xrightarrow[t \text{ é escalar}]{} \tfrac{1}{2} \mathbf{h}^T H \ \xi \ \mathbf{h} \geq 0, \forall \mathbf{h} \in \mathbb{R}^n .
\end{aligned} \tag{1.1.42}$$

One can analogously define the local maximizer case of f*, so if* $\xi \in X$ *it's a local maximizer, then* $\xi \in X$ *it needs to be a stationary point, and if h* ξ is negative defined it's going to be a strong local maximizer.

It's easy to show that if It's stationary, so

1. if H ξ has positive and negative eigenvalues, ξ it is neither a local minimizer nor a local maximizer;
2. if H ξ is semi-defined positive (negative) it may or may not be a local minimizer (maximizer).

Figure 1 Representation and Maximum and Minimum



1.4.1 Directional Derivative

Resolution 20:(directional derivative). Be $f \in C^m(X)$ $\mathbf{x} \in X \subseteq \mathbb{R}^n$ and be $\mathbf{y} \in \mathbb{R}^n$ Where $\|\mathbf{y}\| = 1$. The morder directional derivative of *f in x* in ***the*** y-direction is given by:

$$f^{(m)}(\mathbf{x};\mathbf{y}) \equiv \left. \frac{d^m f(\mathbf{x} + \tau\mathbf{y})}{d\tau^m} \right|_{\tau=0} \tag{1.1.43}$$

The directional derivative of f can be computed by the differentiation chain rule: thus, if *f* if you have $f \in C^1(X)$

$$\begin{aligned} f^{(1)}(\mathbf{x};\mathbf{y}) &= \left. \frac{df}{d\tau}(x_1 + \tau y_1, x_2 + \tau y_2 + \cdots + x_n + \tau y_n) \right|_{\tau=0} \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(x_1, x_2, \cdots, x_n)\, y_i = \mathbf{g}^T(\mathbf{x})\,\mathbf{y} \end{aligned} \tag{1.1.44}$$

If , si has $f \in C^2(X)$

$$f^{(2)}(\mathbf{x};\mathbf{y}) = \mathbf{y}H(\mathbf{x})\,\mathbf{y} \tag{1.1.45}$$

Rewriting the expression of $f^{(1)}(\mathbf{x};\mathbf{y})$ using Cauchy-Schwarz inequality $\left(\left|\mathbf{x}^T\mathbf{y}\right| \le \|\mathbf{x}\|.\|\mathbf{y}\|, \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n \right)$ If you have

$$\begin{aligned} &\max_{y,\|y\|=1} \left| f^{(1)}(\mathbf{x};\mathbf{y}) \right| = \left| f^{(1)}(\mathbf{x};\psi) \right| = \left| \mathbf{g}^T(\mathbf{x})\,\psi \right| \le \left\| \mathbf{g}^T(\mathbf{x}) \right\|.\|\psi\|, \\ &\quad \text{mas } \|\psi\| = 1 \text{ e } \left\|\mathbf{g}^T(\mathbf{x})\right\| = \|\mathbf{g}(\mathbf{x})\| \rightarrow \\ &\quad \left| f^{(1)}(\mathbf{x};\psi) \right| = \|\mathbf{g}(\mathbf{x})\| \Rightarrow \psi = \mathbf{g}(\mathbf{x}) / \|\mathbf{g}(\mathbf{x})\| \end{aligned} \tag{1.1.46}$$

Where ψ is the y direction where f^1 **x; y** is maximum.

In other words: f^1 x; y = 0 in all directions **y** if and only if , that is, if $g(\mathbf{x}) = 0$ is a stationary point of f.

Similarly: f^2 **x; y** to all directions y if and only if $H(\mathbf{x}) > 0$ (Hessian hue is positive-defined). These conclusions allow us to state two fundamental theorems (which will not be demonstrated here).

Theorem 4:Be $f \in C^1(\mathrm{X})$. If $\xi \in \mathrm{X}$ is a local minimizer of f So f^1 **x; y = 0** in all directions **y** .

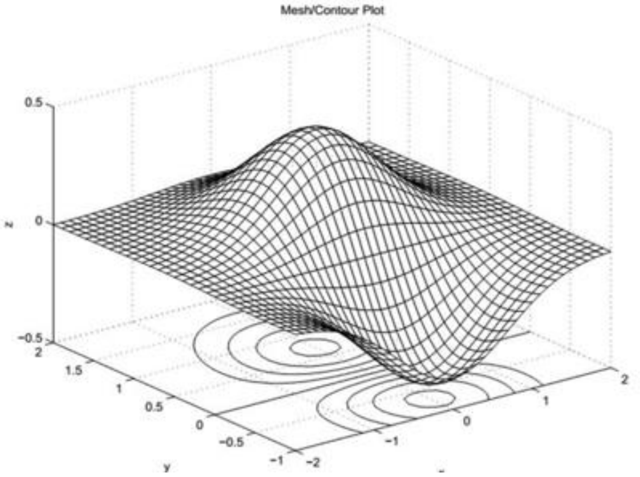
Theorem 5: Let $f \in C^2(\mathrm{X})$. Suppose that for some of you $\xi \in \mathrm{X}$ if you have $f \in C^1(\mathrm{X}) = 0$ in all directions **y**. So $\xi \in \mathrm{X}$ it's a strong local minimizer of f if f^2 $(\mathbf{x}, \mathbf{y}) > 0$ in all directions **y** .

Resolution 21: Let f set on $\mathrm{X} \subseteq \mathbb{R}^n$ and let be $k \in R_f$, where Rf is an interval of f; Then the set

Using Matlab to generate contour lines, you can write (for example):

```
% plots  de uma função de 2 variáveis
x = -2:0.1:2; y = -1:0.1:2;
[X Y] = meshgrid(x,y);
Z = Y.*exp(-(X.^2 + Y.^2));
meshc(X,Y,Z),...
title('Mesh/Contour Plot'),xlabel('x'),...
ylabel('y'),zlabel('z')
```

Figure 2: Contours generated by matlab



A known fact is that if it ξ belongs to a surface (or curve) of level Lk , then the gradient vector o $\nabla f\ \mathbf{x}$ or g $\mathbf{x}$ is perpendicular to L_k at ξ and at all points where the direction of the functional increases most rapidly.

Therefore, gradients near a local strong minimizer point outward while those near a local strong maximizer point inward.

In general, numerical methods for finding a strong local minimizer of a functional are most agile when the level surfaces in the vicinity of the minimized are spherical, and are more difficult when they show a marked distortion of the shape of a sphere. Define the distortion measure of L_k of the spherical shape (perfect spherical shape: $D_k = 1$) as the quantity

$$D_k = \inf_{\xi \in S_k} \left\{ \frac{\sup_{x \in L_k} \|x - \xi\|}{\inf_{x \in L_k} \|x - \xi\|} \right\} \geq 1 \tag{1.1.47}$$

Where S_k is the set of all points inside L_k .

2 VARIATIONAL METHOD IN PROBLEMS WITH FIXED BOUNDARIES

2.1 PRELIMINARES

Equation Section (Next)Variational calculus aims fundamentally to investigate the maxima and minima of the functionals and is very similar to the investigation of maxima and minima of functions. Therefore, in this section, we seek to introduce the fundamental concepts and the main properties of the variational method.

A functional is usually represented by an integral of the type:

$$J\ f\ = \int_a^b I\ f,\ldots,x\ dx \tag{2.1.1}$$

Definition 1: It is called variation or increment δf of the functional argument Jf, unlike two functions F1 X and F 0 X , belonging to the same class of functions M, considered for the functional Jf :

$$\delta f = f_1\ x\ - f_0\ x \tag{2.1.2}$$

For a class of Ck functions : $[a,b]$ i.e., k times differentiable, we obtain

$$\delta f^{\ k} = \delta f^{\ k} \tag{2.1.3}$$

Definition 2: The functions f1 x and f0 x defined in a range [a, b] are said to be close to zero order or null if in the definition range there is if

$$|\delta f| = |f_1\ x\ - f_0\ x\ | << 1 \tag{2.1.3}$$

In general, these functions are said to be close to order k, if:

$$\begin{gathered} |\delta f| = |f_1\ x\ - f_0\ x\ | << 1 \\ |\delta f| = |f_1'\ x\ - f_0'\ x\ | << 1 \\ |\delta f| = |f_1''\ x\ - f_0''\ x\ | << 1 \\ \ldots \\ |\delta f| = |f_1^{\ k}\ x\ - f_0^{\ k}\ x\ | << 1 \end{gathered} \tag{2.1.4}$$

And so if F1 x and f0 x are close to order k, they are close to any order $j \leq k$

2.1.1 Distance between functions of a functional

Definition 3: It's called the distance between $f_0\ x\ \ e f_1\ x\ \in C\ [a,b]$, with $x \in [a,b]$ to the metric defined in this space by:

$$\rho\ f_0,f_1\ = \max_{a<x<b} |f_1\ x\ - f_0\ x\ | \tag{2.1.5}$$

Generally, it is called the nth order distance between $f_0\ x$ e $f_1\ x \in C_k[a,b]$
With $x \in [a,b]$ the highest of the maximums of expressions of type

$$\max_{a<x<b} \left| f_1^{\ k}\ x \ - f_0^{\ k}\ x \ \right|, \forall k = 1,2,\ldots,n \tag{2.1.6}$$

or

$$\rho_n\ f_0, f_1 \ = \max_{0<k<n} \left[\max_{a<x<b} \left| f_1^{\ k}\ x \ - f_0^{\ k}\ x \ \right| \right] \tag{2.1.7}$$

2.1.2 Neighbourhood

Definition 4: The function f_0 x with the function is called € -nth order neighborhood (see item 1.3.6) wit $x \in [a,b]$ to the set f_i of functions whose nth order distance from them to f_0 x are less than €:

$$\rho_n\ f_0\ x\ , f_1\ x \ < \varepsilon \tag{2.1.8}$$

If the €-neighborhood is of zero order, it says that it is a strong €-neighborhood of f_0 x while the first-order €-neighborhoods is called a weak €-neighborhood.

Definition 5: A functional $J\,f$ defined in a M of functions, is called continuous in $f^* = f^*\ x$
In the sense of the proximity of nth order, if whatever the number is, $\varepsilon > 0$ there is a number $\delta > 0$, , such that inequality

$$\left| J\ f\ - J\ f^* \ \right| < \varepsilon \tag{2.1.9}$$

is fulfilled for all permissible functions $f\,x$, i.e. for all functions of meet the conditions

$$\begin{gathered} \left| f\,x\ - f^*\ x\ \right| < \delta \\ \cdots \\ \left| f^{\ n}\ x\ - f^{*\ n}\ x\ \right| < \delta \end{gathered} \tag{2.1.10}$$

In other words, one always has to $\left| J\ f\ - J\ f^* \ \right| < \varepsilon$:

$$\rho_n\ f, f^* < \delta \tag{2.1.11}$$

Example:

Demonstrate that the functional $J(f) = \int_0^1 (y + 2y')dx$ considered in space C1[0,1] is continuous in the *function* f^* x=x in the direction of first-order proximity.

Solution: Let $\varepsilon > 0$ it be and we will show that there is a number $\delta > 0$ such that

$$|J\ f\ - J\ f^*\ | < \varepsilon \tag{a}$$

Whenever

$$\begin{aligned} |f\ x\ - f^*\ x\ | &= |f\ x\ - x| = |y - x| < \delta \\ |f'\ x\ - f^{*'}\ x\ | &= |f'\ x\ - 1| = |y' - 1| < \delta \end{aligned} \tag{b}$$

Like this

$$|J(f) - J(x) = \left| \int_0^1 y + 2y' - x - 2\ dx \right| \le \int_0^1 |y - x| dx + 2\int_0^1 |y' - 1| dx$$

Be it then $\delta = \varepsilon/3$. Therefore, for all functions $f\ x \in C_1[0,1]$ such that conditions (b) are satisfied with $\delta = \varepsilon/3$. condition (a) will also be satisfied, i.e.

$$J(f) - J(x) \le \int_0^1 |y - x| dx + 2\int_0^1 |y' - 1| dx = \frac{\varepsilon}{3} + 2.\frac{\varepsilon}{3} = \varepsilon .$$

2.1.3 Functional increase

Definition 6: Given a functional $J\ \ f$ defined on a class M of functions $f\,x$, the magnitude:

$$\Delta J = \Delta J\ f\ = J\ f + \delta f\ - J\ f \tag{2.1.12}$$

is called the functional increment corresponding to the increment δf the argument.

Definition 7: If the functional increment ΔJ of $J f$ can be represented in the form

$$\Delta J = L\ f\ x\ ,\delta f\ + \beta\ f\ x\ ,\delta f\ \|\delta f\| \quad (2.1.13)$$

where L is a linear functional with respect to δf e $\beta \to 0$ when $\|\delta f\| \to 0$, then the linear increment with δf respect to i.e., , *$L\ f\ x,\ \ \delta f$ is called variation of the functional and is represented by* δJ . In this case si says that the functional $J\ f$ is differentiable at the point $f\, x$.

Definition 8: Let be a linear functional $J f,\ x$ dependent on the elements $f\ x$ and x

$$J = J\ f,x\ = \int_a^b I\ f,x\ dx \quad (2.1.14)$$

2.1.4 Bilinear shape

Definition 9: J is said to be a bilinear form if it is a linear functional with respect to f for fixed x *and a functional for* x with *fixed* $f\, x$, i.e.

$$\begin{aligned} J\ \ f,\alpha x_1 + \beta x_2\ &= \alpha J\ \ f,x_1\ + \beta J\ \ f,x_2 \\ J\ \ \alpha f_1 + \beta f_2,x\ &= \alpha J\ f_1,x\ + \beta J\ f_2,x \end{aligned} \quad (2.1.15)$$

2.1.5 Quadratic Functional

Resolution 10:A Linear Functional $J\ x,x$ Dependent on the elements x and x , i.e. a bilinear form with $f\, x = x$, it's called Quadratic Functional.

Definition 11: A quadratic functional is said to be a positive functional defined if $J\ x,x > 0$ whatever element x is non-zero.

Resolution 12:A Linear Functional $J f$ defined on a normed linear space has second variation if its increment

$$\Delta J = J\ f + \delta f\ - J\ f \quad (2.1.16)$$

can be represented in the form

$$\Delta J = L_1 \; \delta f \; + \frac{1}{2} L_2 \; \delta f \; + \beta \| \delta f \|^2 \qquad (2.1.17)$$

where L_1 is a linear functional, L_2 a quadratic functional, and $\beta \to 0$ when . $\|\delta\| \to 0$ The L_2 δf functional is called the <u>second variation</u> or second differential of the J *f* functional and is called $\delta^2 J$ Quadratic functionals will be studied in more detail in a later section.

2.1.6 Functional extremes

Resolution 13:It is said to be a functional *J f* Reaches its maximum in the f_0 *x* function If the values it takes the functional *J f* in any function close to f_0 *x* are not greater than *J f* That is

$$\Delta J = J \; f \; x \; - J \; f_0 \; x \; \leq 0 \qquad (2.1.18)$$

How $\Delta J \leq 0 \to \Delta J = 0 \Leftrightarrow f = f_0$, It is said that *J f* Achieves the strict maximum

for function f0 *x* . Similarly, if you define or <u>minimum</u> of the functional in a function *f0 x* when $\Delta J \geq 0$ for all curves near *f0 x* .

Example

Demonstrate that the functional $J \; f \; = \int_0^1 x^2 + y^2$ dx achieves a strict minimum in the *y x* *function* $y\,x = 0$.

Solution: Whatever it is $y\,x \in [0,1]$ There is

$$\Delta J = J(f) - J(0) = \int_0^1 (x^2 + y^2)dx - \int (x^2 + 0^2)dx = \int y^2 dx \geq 0$$

And what's more $\Delta J = 0 \Leftrightarrow y \; x = 0.$

Resolution 14:It is said to be a functional *J f* achieves your <u>*Strong relative maximum*</u> in the *f0* if $J \; f \; \leq J \; f_0$ in all permissible functions *f* belonging to a ε- Null-order neighborhood of the function *f0*. Similarly, if define <u>*Strong Relative Minimum*</u> of a functional when $J \; f \; \geq J \; f_0$

Resolution 15:It is said to be a functional *J f* achieves your <u>*Weak relative maximum*</u> or weak in . $J \; f \; \leq J \; f_0$ function *f0* if in all permissible functions *f* belonging to a ε- First-order function neighborhood *f0*. Similarly, if define <u>*Weak relative minimum*</u> or weak of a functional when $J \; f \; \geq J \; f_0$.

The maxima and minima (strong and weak or weak) of a functional $J\,f$ are called relative extremes. The extreme referring to the totality of the functions in which the functional is defined is called the absolute extreme.

Every absolute extreme is at the same time a strong and weak relative extreme, but not every relative extreme will be an absolute extreme.

Example

Let the functional $J\,f = \int_0^{\pi} y^2 \; 1 - y'^2 \; dx$, in the function space $y = f \; x \; \in C_1[0, \pi]$ that satisfy the conditions $y \; 0 \; = y \; \pi \; = 0$.

On the x-axis segment $[0, \pi]$ it has a faint minimum of $J\,f$ that is $J \; f \; \geq J \; f_0$ in a ε- first-rate neighborhood. In fact, it has been $J = 0$ if, $y = 0$ on the other hand, the functions belonging to ε- first-order neighborhood, if has $|y'| < 1$, so that the integrative is positive for $y = 0$ and therefore the functional cancels out only if $y = 0$. That is, the functional reaches a weak minimum in the curve $y \neq 0$.

Theorem 1: Necessary Condition of Functional Extreme. If the differentiable functional $J\,f$ reaches its extreme value on a curve , $f = f_0$ where $f0$ is an inner point of the functional definition field, then in $f = f_0$ we have

$$\delta J \; f_0 \; = 0 \qquad (11.2.19)$$

The functions for which $\delta J = 0$, are called *stationary functions*.

2.1.6.1 Exercises

1. Determine the order of proximity of the curves below:

$$y \; x \; = \cos\left(\frac{nx}{n^2 + 1}\right) \quad \text{e} \quad y_1 \; x \; = 0 \in [0, 2\pi]$$

$$y \; x \; = sen\left(\frac{x}{n}\right) \quad \text{e} \quad y_1 \; x \; = 0 \in [0, \pi]$$

2. Determine the distance between the curves below:

$$y \; x \; = x^{e = x} \quad \text{e} \quad y_1 \; x \; = 0 \in [0, 2]$$

$$y \; x \; = sen \; 2x \quad \text{e} \quad y_1 \; x \; = 0 \in \left[0, \frac{\pi}{2}\right]$$

3. Find the first-order distance between curves $y(x) = \ln(x)$ and $= x$ in the segment $\left[e^{-1}, e\right]$.

4. Analyze the continuities of the following functionalities:

$J(f(x)) = f(x_0) \rightarrow f(x) \in C[a,b] \wedge x \in [a,b]$ in the sense of proximity of null order.

$J(f(x)) = \int_0^1 |y'| dx, y \in C[0,1]$, in the sense of zero-order and first-order proximity.

5. Analyze whether the following functionalities are differentiable:

$$J(f(x)) = f(a) \in C[a,b]$$
$$J(f(x)) = f(a) \in C_1[a,b]$$
$$J(f(x)) = |f(a)| \in C[a,b]$$

6. For the functionals below, determine, in the corresponding spaces, their variations.

$$J(f(x)) = \int_a^b (x + y)\, dx$$

$$J(f(x)) = \int_0^x y' sen(x)\, dx$$

2.1.7 Elementary Problem of Variational Calculus

Be the functional one

$$J(f) = \int_a^b I(f, f_x, x) dx \tag{11.2.20}$$

Where $f \in [a,b] \wedge f_x = \frac{df}{dx} \in [a,b]$. The elementary problem of variational calculus is to find the function that offers the extreme weakness to the functional (11.2.20) and that satisfies the boundary conditions:

$$\begin{aligned} f(a) &= f_a \\ f(b) &= f_b \end{aligned} \tag{11.2.21}$$

Let $f(x)$*, by hypothesis, be the solution function of the problem, and* let h(x) be *another function which differs from* f(x) *a certain quantity:*

$$\delta f(x) = \varepsilon \eta(x) \tag{11.2.22}$$

Where $h(x) \in C_1[a,b]$, ε is a continuously varying parameter, and is $\eta(x)$ an arbitrary function that satisfies the following boundary conditions:

$$\eta\ a\ = \eta\ b\ = 0$$

(11.2.23)

From (11.2.22) and the figure below, we have

$$h\ x\ = f\ x\ + \varepsilon\eta\ x \tag{11.2.24}$$

In this way, you can rewrite the functional (11.2.20) to the *h x function*, such as:

$$J(h) = \int_a^b I\ \ h, h_x, x\ \ dx \tag{11.2.25}$$

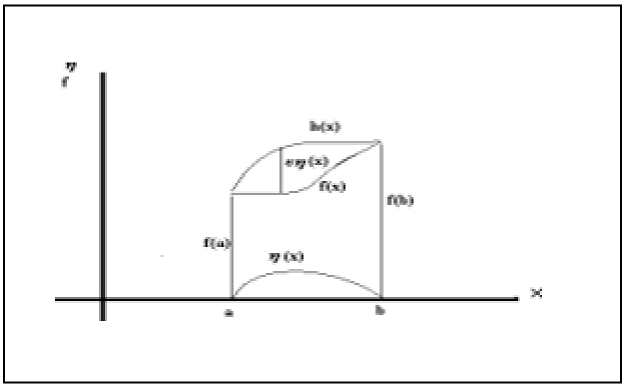


FIGURA 2.1

Where from (11.2.24) can be seen

$$h_x = \frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \varepsilon\frac{\partial \eta}{\partial x} \tag{11.2.26}$$

Since the value of the functional *J varies continuously with , by Taylor's formula, one can develop* J *, as well* as

$$J\ \ h\ \ = J\ \ f\ \ + \varepsilon\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{1}{2!}\varepsilon^2\left.\frac{\partial^2 J}{\partial \varepsilon^2}\right|_{\varepsilon=0} + \cdots \tag{11.2.27}$$

or in variational notation

$$J\ h\ \ = J\ \ f\ \ + \delta J\ \ f\ \ + \frac{1}{2!}\delta^2 J\ \ x\ \ + \cdots \tag{11.2.28}$$

Where $\delta J\ \ f\ \ = \varepsilon\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0}$ e $\delta^2 J\ \ f\ \ = \varepsilon^2\left.\frac{\partial^2 J}{\partial \varepsilon^2}\right|_{\varepsilon=0}$

It is known that the stationarity condition of a functional is $\delta J\ \ f\ \ = 0$ $\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$ either . Therefore, differentiating (11.2.25), we have:

$$\delta J = \left(\frac{dJ}{d\varepsilon}\right)_{\varepsilon=0} = \left\{\int_a^b \frac{dI}{d\varepsilon}dx\right\}_{\varepsilon=0} = \left\{\int_a^b \left(\frac{\partial I}{\partial h}\frac{\partial h}{\partial \varepsilon} + \frac{\partial I}{\partial h_x}\frac{\partial h_x}{\partial \varepsilon}\right)dx\right\}_{\varepsilon=0} \tag{11.2.29}$$

From (11.2.23) and (11.2.26) we have to

$$\frac{\partial h}{\partial \varepsilon} = \eta \quad x$$
$$\frac{\partial h_x}{\partial x} = \eta_x \quad x \tag{11.2.30}$$

Taking (11.2.30) in (11.2.29) is

$$\left(\frac{dJ}{d\varepsilon}\right)_{\varepsilon=0} = \left\{\int_a^b \left(\frac{\partial I}{\partial h}\eta + \frac{\partial I}{\partial h_x}\eta_x\right)dx\right\}_{\varepsilon=0} = \int_a^b \left(\frac{\partial I}{\partial f}\eta + \frac{\partial I}{\partial f_x}\eta_x\right)dx = 0 \tag{11.2.31}$$

Because $\left(\frac{\partial I}{\partial h}\right)_{\varepsilon=0} = \frac{\partial I}{\partial f}$ e $\left(\frac{\partial I}{\partial h_x}\right)_{\varepsilon=0} = \frac{\partial I}{\partial f_x}$

Integrating the second term of the equation (11.2.31) in part obtains

$$\left(\frac{dJ}{d\varepsilon}\right)_{\varepsilon=0} = \int_a^b \left(\frac{\partial I}{\partial f}\eta - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)\eta\right)dx + \left|\frac{\partial I}{\partial f_x}\eta\right|_a^b = 0 \tag{11.2.32}$$

The last term of the right-hand side of (11.2.32) is null and void since what $\eta \; a \; = \eta \; b \; = 0$ is true and valid for any and all permissible functions. Thus equation (11.2.32) becomes:

$$\left(\frac{dJ}{d\varepsilon}\right)_{\varepsilon=0} = \int_a^b \left(\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)\right)\eta dx = 0 \tag{11.2.33}$$

The Basic Motto **of Variational Calculus** is now enunciated:

2.1.7.1 Basic Motto of Variational Calculus

Motto: If *a and* $b \; b > a$ are fixed constants and $G\,x$ is a continuous function belonging to $C\,a,b$

What if

$$\int_a^b \eta \;\; x \;\; G \;\; x \;\; dx = 0 \tag{11.2.34}$$

for every continuously differentiable function $\eta \;\; x$ that satisfies the conditions $\eta \;\; a \; = \eta \;\; b \; = 0$ one can conclude that $G\,x \; = 0$ for the entire interval $a. \leq x \leq b$

From the basic motto and from (11.2.33) it can be seen that

$$\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right) = 0 \tag{11.2.35}$$

which is the *Euler equation or* Euler-Lagrange *equation*. The integral curves of Euler's equation are called extremes or Lagrange curves.

2.1.7.2 Euler–Lagrange equation

Theorem 1: The necessary condition for the functional

$$J(f) = \int_a^b I\ \ f, f_x, x\ \ dx \tag{11.2.36}$$

defined in the set $C_1\left[a,b\right]$ and that satisfy the boundary conditions $f_1 = f\ a \quad f_2 = f\ b$ and , reach its extreme value in the $f\,x$ is that this function checks the Euler-Lagrange equation (11.2.35):

$$\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right) = 0.$$

This condition is necessary for the weak extreme; But as every strong extreme is also at the same time weak, any condition for the weak extreme will also be a condition for the strong extreme.

Example:

Be the functional one $J\ f\ = \int_1^2 y'^2 - 2xy\ \ dx \therefore y\ 1\ = 0 \wedge y\ 2\ = -1.$

The Euler–Lagrange equation will be $y'' + x\ = 0$. Solving the differential equation (Euler–Lagrange eq.*), we have*

$$y = -\left(\frac{x^3}{6}\right) + \alpha x + \beta$$

Using the boundary conditions, we find, $\alpha = \frac{1}{6}\ e\ \ \beta = 0.$

Therefore, the extreme can be reached in the function: $y = \left(\frac{x}{6}\right)\ 1 - x^2\ \ .$

2.1.7.3 Exercises

Find the Euler–Lagrange equation and the function in which the given functional can reach extreme, to:

$$1.\ J\ f\ =\int_0^{\frac{\pi}{2}} f'^2+f^2\ dx\ \therefore f\ 0\ =0\wedge f\left(\frac{\pi}{2}\right)=1.$$

$$2.\ J\ f\ =\int_1^3 3x-f\ fdx\ \therefore f\ 1\ =1\wedge f\ 3\ =4\frac{1}{2}.$$

$$3.\ J\ f\ =\int_0^{2\pi} f'^2+f^2\ dx\ \therefore f\ 0\ =f\ 2\pi\ =1.$$

Theorem 2: Let $f\,x$ be the solution of the Euler–Lagrange equation. If the function $I\ x,f,f_x$ has continuous partial derivatives up to and including the second order, then the function $f=f\ x$ has a second continuous derivative at all points x,y for which

$$\frac{\partial^2 I}{\partial f_x^2}\neq 0$$

Theorem 3: (Bernstein). If in the equation $f_{xx}=I\ x,f,f_x$ the functions $I\ \frac{\partial I}{\partial f}$ are continuous at every finite point x, f, for any finite value of f_x and there is still a constant $k>0$ and functions

$$\alpha=\alpha\ x,f\ \geq 0 \qquad (11.2.38)$$
$$\beta=\beta\ x,f\ \geq 0$$

limited in any finite portion of the plane, such that

$$\frac{\partial I}{\partial f}=I_f>k \qquad (11.2.39)$$
$$\left|I\right|\leq \alpha f_x^2+\beta$$

Then, through any two points of the plane x_1,f_1 x_2,f_2 and of different abscissas $x_1\neq x_2$ passes one and only one integral curve $f=\varphi\ x$ of (11.2.39).

Examples

1. Demonstrate that through any two points of the plane of distinct abscissas passes a single extreme function of the functional

$$J(y)\qquad e^{2y^2}(y'^2\quad 1)dx$$

Euler's equation for the above functional is: $y''=2y\ 1+y'^2$ if it is in the format (11.2.37) of Bernstein's theorem, then one can apply it. In fact, we have:

$$f_{xx} = I\ x,y,y' = 2y\ 1+y'^2$$
$$\frac{\partial I}{\partial f} = \frac{\partial I}{\partial y} = I_y = 2\ 1+y'^2$$

In the expression $of\ Iy$, it is seen that whatever the value y', y'^2 of is positive, $I_y \geq 2 = k$.
then in addition, $|I| = |2y\ 1+y'^2| \leq 2|y| + 2|y|y'^2$
Comparing it with the second expression of (11.2.39) of Bernstein's theorem, we see that:

$$\alpha = 2|y| > 0$$
$$\beta = 2|y| > 0$$

and that also satisfies the condition (11.2.38), passes a single extreme function *xy*
J y.
2. Demonstrate that there is no end of the functional $I(y) = \int y^2 + \sqrt{1+y'^2}\,dx$ that passes through any two points of the plane of distinct abscissas.
Solution
Euler's equation for the given functional has the form:

$$y'' = 2y\ 1+y'^2\ ^{3/2}$$

Analyzing the above expression, it can be seen that Bernstein's theorem can be applied, since the expression fulfills the extreme condition (11.2.39) of the given functional.
So let there be any two dots, for example $0,0$ and $1/2, 2$. Rewriting Euler 's equation of the given functional, making $y' = p; y'' = p\frac{dp}{dy}$.

$$p\frac{dp}{dy} = 2y\ 1+p^2\ ^{3/2}$$

Separating the variables, we have: $p\ 1+p^2\ ^{3/2}\,dp = 2ydy$ that by integrating we have

$$-\ 1+p^2\ ^{-1/2} = y^2 - C \Rightarrow C - y^2\ \ 1+p^2\ ^{1/2} = 1$$

So

$$\frac{dx}{dy} = C - y^2\ ^{-1}\left[1 - C - y^2\ ^2\right]^{1/2}$$

where C is a real constant. Separating the variables and integrating from *(0,0)* to *(1/2,2)*, we obtain:

$$\frac{1}{2} = \int_0^2 \left(\frac{C - y^2}{\sqrt{1-(C-y^2)^2}} \right) dy$$

Looking at the above expression, it can be seen that whatever the real C is, the denominator will be complex in a certain interval of $a,b \subset 0,2$ the variation of y . Therefore equality is impossible. This means that no extreme can be drawn from the points considered, so Bernstein's theorem fails here.

2.1.7.4 Exercise

The equation of an ellipse is known to be derived from the condition that the sum of the distances from any point A on it to two other fixed points $F_1(x_1=-c, y=0)$ e $F_2(x_2=+c, y=0)$ is constant. Find the tangent and the normal to an ellipse at any point A. Find the angles between the F_1A, F_2A, and normal lines at point A.

2.1.8 Generalizations of the Elementary Problem of Variational Calculus

Let us now find the Euler–Lagrange equation for the functional of type

$$J = \int_a^b I \ f, f_x, f_{xx}, x \ dx \qquad (11.2.40)$$

where f, f_x, f_{xx} are the dependent variables. Be then

$$\begin{aligned} f_a &= f \ a \\ f_b &= f \ b \\ f_{xa} &= f_x \ a \\ f_{xb} &= f_x \ b \end{aligned} \qquad (11.2.41)$$

known values. Here, in this case, not Only η (ou $\delta f = \varepsilon\eta$) is it zero in a and b, but also η_x ($\delta f_x = \varepsilon\eta_x$) is null at the contour points, so

$$\delta J = \int_a^b \left(\frac{\partial I}{\partial f}\delta f + \frac{\partial I}{\partial f_x}\delta f_x + \frac{\partial I}{\partial f_{xx}}\delta f_{xx} \right) dx = 0 \qquad (11.2.42)$$

Being $\delta f_{xx} = \varepsilon\eta_{xx}$. Integrating the last two terms in order to place under the sign of the integral only the terms multiplied by δf, we have:

$$\int_a^b \left(\frac{\partial I}{\partial f_x}\delta f_x \right) dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial I}{\partial f_x} \right) \delta f_x dx + \frac{\partial I}{\partial f_x}\delta f \Big|_a^b \qquad (11.2.43)$$

and

$$\int_a^b \left(\frac{\partial I}{\partial f_{xx}} \delta f_{xx} \right) dx = \int_a^b \frac{d^2}{dx^2} \left(\frac{\partial I}{\partial f_{xx}} \right) \delta f \, dx - \frac{d}{dx} \frac{\partial I}{\partial f_{xx}} \delta f \Big|_a^b + \frac{\partial I}{\partial f_{xx}} \delta f_x \Big|_a^b \qquad (11.2.44)$$

2.1.8.1 Fourth-order Euler–Lagrange equations

You can then rewrite the expression δJ as follows:

$$\delta J = \int_a^b \left[\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x} \right) + \frac{d^2}{dx^2}\left(\frac{\partial I}{\partial f_{xx}} \right) \right] \delta f dx + \left| \left\{ \frac{\partial I}{\partial f_x} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_{xx}} \right) \right\} \right| \delta f \Big|_a^b + \left| \frac{\partial I}{\partial f_{xx}} \delta f_x \right|_a^b \qquad (11.2.45)$$

since the permissible functions are such that $\delta f = \delta f_x = 0$ in $x = a$ e $x = b$, then the boundary terms, in the above expression, are null and the Euler–Lagrange equation has the following form:

$$\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x} \right) + \frac{d^2}{dx^2}\left(\frac{\partial I}{\partial f_{xx}} \right) = 0 \qquad (11.2.46)$$

which is a fourth-order differential equation.

2.1.8.2 Higher-Order Euler–Lagrange equations

Generally, for functionals of the type:

$$J \; f \; = \int_a^b I \;\; f, f_x, f_{xx}, \cdots, f_{n \; x} \;, x \;\; dx \qquad (11.2.47)$$

for which the function f $x = a$ *and* $x = b$ *the derivatives of* f *will be assumed to be known in* $n - 1$. In this case we have:

$$\delta J = \int_a^b \left\{ \frac{\partial I}{\partial f} \delta f + \frac{\partial I}{\partial f_x} \delta f_x + \ldots + \frac{\partial I}{\partial f_{x(n)}} \delta f_{x(n)} \right\} dx = 0 \qquad (11.2.48)$$

Or

$$\delta J = \int_a^b \left\{ \sum_{k=0}^{k=n} \frac{\partial I}{\partial f_{x(k)}} \delta f_{x(k)} \right\} dx = 0 \qquad (11.2.49)$$

where the different terms must be integrated in parts until the integrating is multiplied only by δf . A typical term for piecewise integration is:

$$\int_a^b \frac{\partial I}{\partial f_{x(j)}}\delta f_{x(j)}dx = \left|\frac{\partial I}{\partial f_{x(j)}}\delta f_{x(j-1)}\right|_a^b - \left|\frac{d}{dx}\frac{\partial I}{\partial f_{x(j-1)}}\delta f_{x(j-2)}\right|_a^b + \left|\frac{d^2}{dx^2}\left(\frac{\partial I}{\partial f_{x(j-2)}}\right)\delta f_{x(j-3)}\right|_a^b + \ldots + (-1)^{j-1}\left|\frac{d^{j-1}}{dx^{j-1}}\left(\frac{\partial I}{\partial f_x}\right)\delta f\right|_a^b + (-1)^j\int_a^b\left\{\frac{d^j}{dx^j}\left(\frac{\partial I}{\partial f_{x(j)}}\right)\delta f\right\}dx \tag{11.2.50}$$

or in a more compact form:

$$\int_a^b \frac{\partial I}{\partial f_{x(j)}}\delta f_{x(j)}dx = \sum_{k=0}^{k=j-1}\left|\left\{(-1)^k\frac{d^k}{dx^k}\left(\frac{\partial I}{\partial f_{x(j-k)}}\right)\delta f_{x(j-k-1)}\right\}\right|_a^b + (-1)^j\int_a^b\left\{\frac{d^j}{dx^j}\left(\frac{\partial I}{\partial f_{x(j)}}\right)\delta f\right\}dx \tag{11.2.51}$$

Euler's equation for functionals of this type then has the following form:

$$\frac{\partial I}{\partial f} - \frac{d}{dx}\frac{\partial I}{\partial f_x} + \cdots + (-1)^j\frac{d^j}{dx^j}\left(\frac{\partial I}{\partial f_{x(j)}}\right) + \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial I}{\partial f_{x(n)}}\right) = 0 \tag{11.2.52}$$

or

$$\sum_{k=0}^{k=n}\left\{(-1)^k\frac{d^k}{dx^k}\left(\frac{\partial I}{\partial f_{x(k)}}\right)\right\} = 0 \tag{11.2.53}$$

Example

Finding the extreme function of the functional:

$$J(y) = \int_0^1 (360x^2y - y''^2)\,dx \therefore y(0) = y(1) = 0 \wedge y'(0) = 1 \wedge y'(1) = 2.5.$$

Solution:

Euler's equation for this problem has the form:

$$\delta J = \frac{\partial I}{\partial y} - \frac{d}{dx}\left(\frac{\partial I}{\partial y_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial I}{\partial y_{xx}}\right) = 0$$

That is

$$\delta J = 360x^2 + \frac{d^2}{dx^2}\left(-2y''\right) = 360x^2 - 2y^{(4)} = 0$$

Hence, $y^{(4)} = 180x^2$ which integrates we have:

$$y = \frac{1}{2}x^6 + \alpha x^3 + \beta x^2 + \chi x + \gamma$$

Taking into account the boundary conditions, we find

$$\alpha = \frac{3}{2}$$
$$\beta = -3$$
$$\chi = 1$$
$$\gamma = 0$$

Therefore, the extreme function sought after is: $y = \frac{1}{2}x^6 + \frac{3}{2}x^3 - 3x^2 + x$.

They are now functional with several functions as dependent variables, which is the case with many engineering problems. Consider the functional

$$J(f,g) = \int_a^b I(f,g,f_x,g_x,x)\,dx \tag{11.2.54}$$

with the following boundary conditions:

$$\begin{aligned} f_a &= f(a) \\ f_b &= f(b) \\ g_a &= g(a) \\ g_b &= g(b) \end{aligned} \tag{11.2.55}$$

Let be $\eta(x)$ $\zeta(x)$ two permissible functions that satisfy the above boundary conditions and are such that they define the following functions:

$$\begin{aligned} h(x) &= f(x) + \varepsilon\eta(x) = f + \delta f \\ z(x) &= g(x) + \varepsilon\zeta(x) = g + \delta g \end{aligned} \tag{11.2.56}$$

With $\eta(a) = \eta(b) = \zeta(a) = \zeta(b) = 0$ and where $f(x)$ *and* g x *are extremal functions* $J(f,g)$ *and* δf *And* δg are their variations. Like so:

$$\delta J = \varepsilon\left(\frac{dJ(f,g)}{d\varepsilon}\right)_{\varepsilon=0} = 0 \tag{11.2.57}$$

that is:

$$\delta J = \varepsilon\left\{\int_a^b\left[\frac{\partial I}{\partial h}\frac{\partial h}{\partial \varepsilon}+\frac{\partial I}{\partial h_x}\frac{\partial h_x}{\partial \varepsilon}+\frac{\partial I}{\partial z}\frac{\partial z}{\partial \varepsilon}+\frac{\partial I}{\partial z_x}\frac{\partial z_x}{\partial \varepsilon}\right]dx\right\}_{\varepsilon=0} \tag{11.2.58}$$

For $\varepsilon = 0$ remembering that $\varepsilon\eta = \delta f$ and $\varepsilon\zeta = \delta g$, we have:

$$\delta J = \varepsilon\left\{\int_a^b\left[\frac{\partial I}{\partial h}\eta+\frac{\partial I}{\partial h_x}\eta_x+\frac{\partial I}{\partial z}\zeta+\frac{\partial I}{\partial z_x}\zeta_x\right]dx\right\} \tag{11.2.59}$$

$$\delta J = \left\{\int_a^b\left[\frac{\partial I}{\partial h}\varepsilon\eta+\frac{\partial I}{\partial h_x}\varepsilon\eta_x+\frac{\partial I}{\partial z}\varepsilon\zeta+\frac{\partial I}{\partial z_x}\varepsilon\zeta_x\right]dx\right\}$$

$$\delta J = \left\{\int_a^b\left[\frac{\partial I}{\partial h}\delta f+\frac{\partial I}{\partial h_x}\delta f_x+\frac{\partial I}{\partial z}\delta g+\frac{\partial I}{\partial z_x}\delta g_x\right]dx\right\}$$

By integrating the terms $\delta f_x, \delta g_x$ in the above expression, we get:

$$\delta J = \int_a^b\left\{\left[\frac{\partial I}{\partial f}-\frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)\right]\delta f+\left[\frac{\partial I}{\partial g}-\frac{d}{dx}\left(\frac{\partial I}{\partial g_x}\right)\right]\delta g\right\}dx + \tag{11.2.60}$$
$$+\left|\frac{\partial I}{\partial f_x}\delta f\right|_a^b+\left|\frac{\partial I}{\partial g_x}\delta g\right|_a^b$$

The last two terms of the above expression are zero due to boundary conditions, and the terms multiplied by δf δg and are also null, since and are δf δg arbitrary. Thus, Euler's equations are:

$$\frac{\partial I}{\partial f}-\frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)=0 \tag{11.2.61}$$
$$\frac{\partial I}{\partial g}-\frac{d}{dx}\left(\frac{\partial I}{\partial g_x}\right)=0$$

Generalizing to shape functionals:

$$J = \int_a^b I\left(f_1, f_2, \cdots, f_n, f_{1_x}, f_{2_x}, \cdots, f_{n_x}, x\right) dx \tag{11.2.62}$$

with the distinct f_i functions and with the following boundary conditions:

$$f_{1_a} = f_1(a); f_{1_b} = f_1(b); \tag{11.2.63}$$
$$\cdots$$
$$f_{n_a} = f_n(a); f_{n_b} = f_n(b);$$

then we get the following system of Euler's equations:

$$\frac{\partial I}{\partial f^k}-\frac{d}{dx}\left(\frac{\partial I}{\partial f^k_x}\right)=0 \therefore k = 1,2,\ldots,n \tag{11.2.64}$$

Examples

1. Find the extremes of the functional $J\left[y(x), z(x)\right] = \int_1^2 \left(y'^2 + z^2 + z'^2\right) dx$ with the following boundary conditions: $y(1) = 1; y(2) = 2; z(1) = 0; z(2) = 1.$

Solution:

Applying the above theory, one finds the following system of ordinary differential equations:

$$\begin{cases} y'' = 0 \\ z - z'' = 0 \end{cases}$$

whose solution is of the type:

$$\begin{cases} y = \alpha x + \beta \\ z = \chi e^x + \kappa e^{-x} \end{cases}$$

Applying the boundary conditions, we have: $\alpha = 1; \beta = 0; \chi = \dfrac{1}{e^2 - 1}; \kappa = -\dfrac{e^2}{e^2 - 1}$

so the extremes request is:

$$\begin{cases} y = x \\ z = \dfrac{\sinh(x - 1)}{\sinh(1)} \end{cases}$$

2. Finding the extremes of the functional $J\left[y(x), z(x)\right] = \int_0^\pi \left(2yx - 2y^2 + y'^2 - z'^2\right) dx$

with the following boundary conditions:

$y(0) = 0; y(\pi) = 1; z(0) = 0; z(\pi) = -1.$

Solution:

Similar to what was done in the previous example, Euler's equations are:

$$\begin{cases} y'' + 2y - z = 0 \\ z'' + y = 0 \end{cases}$$

From this, eliminating the *function* z, we obtain:

$$y^{iv} + 2y'' + y = 0$$

which is an ordinary fourth-order differential equation, the generic solution of which has the form:

$$y = A\cos(x) + B\sin(x) + x\,(C\cos(x) + D\sin(x))$$

Applying the boundary conditions $y(0) = 0; y(\pi) = 1 \rightarrow A = 0 \wedge C = -\frac{1}{\pi}$, so that:

$$y = B\sin(x) + Dx\sin(x) - \frac{x}{\pi}\cos(x)$$

The function z is determined from $z = y'' + 2y$. the condition Thus,

$$z = B\sin(x) + D\,(2\cos(x) + x\sin(x)) + \left(\frac{1}{\pi}\right)(2\sin(x) - x\cos(x))$$

Applying the boundary conditions now $z(0) = 0; z(\pi) - 1$, we get: $D = 0$ and B = any arbitrary number. Like this:

$$z = B\sin(x) + \left(\frac{1}{\pi}\right)(2\sin(x) - x\cos(x))$$

From this it follows that the extremals of the given functional are the family of curves:

$$y = B\sin(x) - \left(\frac{x}{\pi}\right)\cos(x)$$
$$z = B\sin(x) + \left(\frac{1}{\pi}\right)(2\sin(x) - x\cos(x))$$

with *arbitrary* B.

2.1.8.3 Exercises:

Finding the extremes of the functional:

1. $J(y) = \int_0^1 (y^2 + 2y'^2 + y''^2)\,dx$

com $y(0) = y(1) = 0; y'(0) = 1; y'(1) = -\sinh(1)$.

2. $J(x) = \left(\frac{1}{2}\right)\int_0^1 y''^2 dx \therefore y(0) = y'(0) = 0; y'(1) = 1.$

3. $J(y(x), z(x)) = \int_0^{\pi/2} (y'^2 + z'^2 - 2yz)\,dx \therefore y(0) = z(0) = 0; y\left(\frac{\pi}{2}\right) = z\left(\frac{\pi}{2}\right) = 1$

$$4.\ J\ \ y\ \ x\ \ ,z\ x\ \ = \int_0^1 y'^2 + z'^2 - 2y\ \ dx\ \therefore y\ \ 0\ \ = z\ \ 0\ \ = 0; y\ \ 1\ \ = \frac{3}{2}; z\ \ 1\ \ = 1$$

2.1.9 Principle of Least Action

As physics is well known, the success of the universal principle of minimum potential energy used to determine the equilibrium position of a system stimulates the search for a universal analogous principle with the help of which it may be possible to determine the possible motions of a system. This led to the discovery of the *principle of least action*.

Let us first be a special case: suppose a particle of mass m in motion along the x-axis under the action of a force with potential $U\ x$. As is well known from physics, the equation of particle motion is

$$m\frac{d^2x}{dt^2} = -U'\ \ x\ \ \to m\frac{d^2x}{dt^2} + U'\ \ x\ \ = 0 \tag{11.2.65}$$

It is easy to choose a functional for which the last equation is precisely an Euler equation. Denoting can be rewr $\frac{dx}{dt} = \dot{x}$

$$\frac{dU}{dx} + \frac{d}{dt}\ \ m\dot{x}\ \ = 0 \to \frac{d\ \ -U}{dx} - \frac{d}{dt}\left[\frac{d}{dx}\left(\frac{m\dot{x}^2}{2}\right)\right] = 0 \tag{11.2.66}$$

This last form is a reassembly of Euler's equation, which in the case of the desired function $x\ t$ must have the form:

$$\frac{\partial}{\partial x}F\ \ t,x,\dot{x}\ \ - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}F\ \ t,x,\dot{x}\ \ \right] = 0 \tag{11.2.67}$$

As in both the above expression (11.2.67) and the previous one (11.2.66) the function F is derivable on both sides, and considering the sign of the equation above, it can be seen that the derivatives are zero, so that they are

$$F\ \ t,x,\dot{x}\ \ = \left[\frac{m\dot{x}^2}{2} - U\ \ x\ \ \right]$$

(11.2.67), if you have

$$\frac{\partial}{\partial x}\left[\frac{m\dot{x}^2}{2} - U\ \ x\ \ \right] - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left[\frac{m\dot{x}^2}{2} - U\ \ x\ \ \right]\right\} = 0 \tag{11.2.68}$$

So the desired functional is

$$J = \int_{t_1}^{t_2} \left[\frac{m\dot{x}^2}{2} - U\ x \right] dt \qquad (11.2.69)$$

Note that the term $\frac{m\dot{x}^2}{2}$ is precisely the kinetic energy E of the motion of the particle. Thus, denoting the integrative as $L = E - U$, si has the so-called Lagrangian function. Thus, the variational problem consists of looking for the stationary value of the integral:

$$J = \int_{t_1}^{t_2} L dt$$

That is known as the name of "action"; aquit1 and *t2* are the start and end time of the movement. It can be seen that in a large number of cases the interval between *t1* and *t2* is very small, and in this situation the integral above has a minimum value and not merely a stationary value for the real motion. For this reason, the possibility of thinking that the movement through the procedure from the variational problem to the integral above is called the principle of least action.

It is important to note that the variational principle of least action is universal in nature and remains valid for any closed system not involving energy dissipation, e.g., via friction; incidentally, a dissipative system can, in a sense, be considered open. According to this principle, of all the ways devised (under given constraints) of moving from one state at time *t1* to another state at time *t2*, the system chooses the mode for which the action assumes a stationary value (minimum, as a rule).

Here, the Lagrangian function *L* is the difference between the kinetic energy and the potential energy of the system, each of these energies expressed in generalized coordinates of the system and its temporal derivatives. Thus, the principle of least actions is applicable both to systems with a finite number of degrees of freedom and also to continuous media, and not only mechanical, but also electromagnetic and other phenomena.

Examples

Let the application of the principle of least action to determine the equation of the transverse oscillations of a membrane satisfy Laplace's equation, with the condition of contour $z|_{\Omega} = \varphi$ (data) . The kinetic energy of an element $d\sigma$ by membrane is equal to $\frac{1}{2}\rho d\sigma \left(\frac{dz}{dt}\right)^2$, where is the density of the membrane surface, so the total kinetic energy of the membrane is

$$E = \frac{1}{2}\rho \iint_{\sigma} \left(\frac{dz}{dt}\right)^2 dxdy$$

Whereas the voltage in the membrane remains unchanged in the vibration process (voltage T), then the accumulated potential energy is

$$P = \frac{T}{2} \iint_{\sigma} \left[\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] dxdy$$

So the Lagrangian function and the action are:

$$L = \frac{1}{2} \iint_{\sigma} \left\{\rho\left(\frac{dz}{dt}\right)^2 - T\left[\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]\right\} dxdy$$

$$S = \frac{1}{2} \int_{t_1}^{t} \iint_{\sigma} \left\{\rho\left(\frac{dz}{dt}\right)^2 - T\left[\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]\right\} dxdydt$$

Applying Euler's equation with respect to the independent variables (see equations (11.2.65) and (11.2.72)), we find the membrane oscillation equation:

$$-\frac{\partial}{\partial t}\left(\rho \frac{\partial z}{\partial t}\right) + \frac{\partial}{\partial x}\left(T \frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(T \frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \therefore a = \sqrt{\frac{T}{\rho}}$$

Let be a dynamic system such as the pendulum which is characterized by a body of mass m subject to an actuating θ element represented by a rod of length l in which the torque applied to the actuator is u. As is known from physics, the movement of the pendulum is performed in a circular way where a variable $\dot{\theta}$ measures the angular position in relation to the vertical and its derivative represents the angular velocity; there is also a force of gravity acting on the system that acts directly on the mass and a drag that is proportional to the angular velocity.

From the theory of variational calculus and physics, it is worth remembering that, in the case of a dynamically with n degrees of freedom and with generalized coordinates $q \in \mathbb{R}^n$ and with generalized external forces $Q \in \mathbb{R}^n$, this is described by the following Euler–Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L\ \ q, q}{\partial q}\right) - \frac{\partial L\ \ q, q}{\partial q} = Q$$

$$\therefore L = T\ \ q, q\ \ - V\ \ q$$

Given this information, one can model the pendulum, defining what its kinetic energy is, considering that the linear velocity in relation to its angular velocity is $\dot{s} = r\dot{\theta} = l\dot{\theta}$ (r the radius of the circumference of its motion):

$$T = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2$$

On the other hand, the potential energy of the pendulum system is given by the difference in the height of the pendulum in relation to its vertical position, given by:

$$V = mgl\left(1 - \cos\theta\right)$$

The pendulum drag due to my motion in the air is proportional to the loss function of Rayleigh sees to angular velocity, and is given by:

$$F\left(\dot{\theta}\right) = \frac{1}{2}k\left(l\dot{\theta}\right)^2$$

Thus, given that the Lagrangian functional is given by $L = T - V$, then the Euler–Lagrange *equation is given by* :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial F}{\partial \dot{\theta}} = u$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}; \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta; \quad \frac{\partial F}{\partial \dot{\theta}} = kl^2\dot{\theta};$$

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

Hence the equation of the nonlinear model of the actuated pendulum

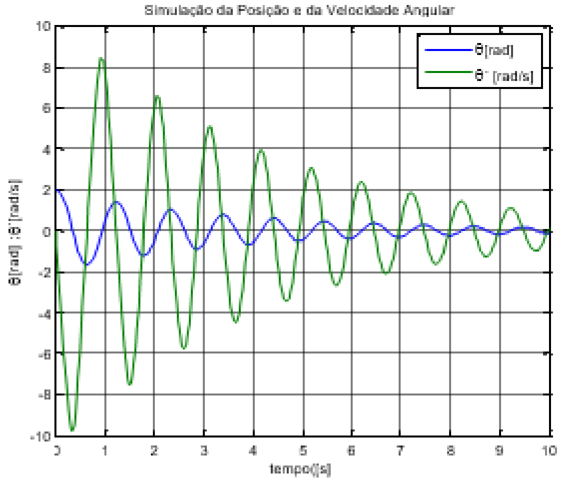
$$ml^2\ddot{\theta} + mgl\sin\theta + kl^2\dot{\theta} = u$$

Below is a program to solve the equation of the nonlinear model:

```
function pendulo(L,M,U)
% ====================================================
% Problema: pêndulo de comprimento L com massa M
% entrada:    L = comprimento da haste do pendulo [m]
%             M = massa do pendulo em [kg]
%             U = torque aplicado no atuador [Nm]
% ====================================================
if nargin == 0
    L=0.25; M=0.2; U=0.0;
end
clc; global k l m u g;
%
k = 0.1; % coeficiente de Rayleigh
if L==0
    l = 0.25; % comprimento da haste do pendulo em [m]
else
    l=L;
end
if M==0
    m = 0.20; % massa do pendulo em [kg]
else
    m = M;
end
if U == 0
    u = 0 ;   % torque aplicado no atuador em [Nm]
else
    u = U;
end
g    = 9.8; % aceleração da gravidade [m/s2]
x0   = [2 0]; % condição inicial = \theta=2 radiano;\theta´=0
tsim = [0 10];  % tempo de simulação entre 0 e 10 segundos
%
%options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-5]);
[t,y] = ode45(@naolinear,tsim,x0);
figure
plot(t,y,'-'),grid on
title('Simulação da Posição \theta e da Velocidade Angular \theta´')
xlabel('tempo([s]')
ylabel('\theta[rad] ;\theta´[rad/s]')
```

```
legend('\theta[rad]','\theta´ [rad/s]');
end
%
%%
function xdot = naolinear(t,x)
global k l m u g;
xdot = [x(2); -(g/l)*sin(x(1))-(k/m)*x(2)+u/(m*l*l) ];
end
```

Running this program you have the following graphic output for the position θ and angular velocity $\dot{\theta}$:



2.1.9.1 Functionals That Depend on Functions of Multiple Independent Variables

Be the functional one

$$J = \iint_{\Omega} I(f, f_x, f_y, x, y)\,dx\,dy \quad (11.2.70)$$

for which we want to analyze the extreme, and whose boundary conditions on the border $\partial\Omega$ of the region Ω are known.

Be $\eta\ x,y$ an admissible function that satisfies the boundary conditions and has continuous derivatives to the desired degree and that

$$h\ x,y = f\ x,y + \varepsilon\eta\ x,y = f + \delta f$$
$$h_x\ x,y = f_x\ x,y + \varepsilon\eta_x\ x,y = f_x + \delta f_x$$
$$h_y\ x,y = f_y\ x,y + \varepsilon\eta_y\ x,y = f_y + \delta f_y$$

Like this

$$\delta J = \varepsilon\left(\frac{dI}{d\varepsilon}\right)_{\varepsilon=0} = 0 \quad (11.2.71)$$

$$\delta J = \varepsilon\left(\iint_{\Omega}\left\{\frac{\partial I}{\partial h}\frac{\partial h}{\partial \varepsilon} + \frac{\partial I}{\partial h_x}\frac{\partial h_x}{\partial \varepsilon} + \frac{\partial I}{\partial h_y}\frac{\partial h_y}{\partial \varepsilon}\right\}dxdy\right)_{\varepsilon=0} = 0 \quad (11.2.72)$$

$$\delta J = \varepsilon\left(\iint_{\Omega}\left\{\frac{\partial I}{\partial h}\delta f + \frac{\partial I}{\partial h_x}\delta f_x + \frac{\partial I}{\partial h_y}\delta f_y\right\}dxdy\right) = 0 \quad (11.2.73)$$

Integrating the second and third terms of the above expression into Green's Theorem, we obtain:

$$\iint \frac{\partial I}{\partial f_x}\delta f_x dxdy = -\iint \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right)\delta f dxdy + \oint \frac{\partial I}{\partial f_x}\delta f \frac{dy}{ds}ds$$
$$\iint \frac{\partial I}{\partial f_y}\delta f_y dxdy = -\iint \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_y}\right)\delta f dxdy + \oint \frac{\partial I}{\partial f_y}\delta f \frac{dy}{ds}ds \quad (11.2.74)$$

Taking these results into δJ, we have:

$$\delta J = \iint_{\Omega}\left\{\frac{\partial I}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial f_y}\right)\right\}\delta f.dx.dy + \quad (11.2.75)$$
$$+\oint_{\partial\Omega}\left\{\frac{\partial I}{\partial f_x}\frac{dy}{ds} - \frac{\partial I}{\partial f_y}\frac{dx}{ds}\right\}\delta f.ds = 0$$

Or, since $\delta f = \varepsilon\eta$ satisfies the boundary conditions in $\partial\Omega$, the second integral of the above expression is zero, so the stationarity condition is:

$$\delta J = \iint_{\Omega}\left\{\frac{\partial I}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial f_y}\right)\right\}\delta f.dx.dy = 0 \quad (11.2.76)$$

Since the variation of δf is arbitrary (a δf imposes only general constraints on continuity and derivability, annulment in the boundary, etc.) and the first factor of the integral above is continuous, then from the Basic Lemma the function $f\ x{,}y$ that performs extreme in the given functional is:

$$\frac{\partial I}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial f_y}\right) = 0 \quad (11.2.77)$$

This second-order differential equation in partial derivatives (11.2.77), which is the Euler equation of the functional (11.2.70), is given the particular name of *the Euler–Ostrogradski* equation.

Examples of this type of functional are:

$J = \iint f_x^2 + f_y^2$ dxdy whose corresponding *Euler–Ostrogradski* equation is of the form $\nabla^2 f = 0$, which is Laplace's equation. In addition, finding a continuous solution of this equation in Ω, with contour values in known $\partial\Omega$, is a problem called the *Dirichlet* problem, which is fundamental to mathematical physics and engineering; it is a typical stationary heat conduction problem.

whose corresponding *Euler–Ostrogradski* equation is of the form , which is the Poisson equation.

$J = \iint f_x^2 + f_y^2 + 2f.p\ \ x{,}y\ \ dxdy$ whose corresponding Euler-Ostrogradski equation is of the form $\nabla^2 f = p\ \ x{,}y$, which is Poisson's equation.

$J = \iint \sqrt{1 + f_x^2 + f_y^2}\, dxdy$ which is the functional whose minimum solves the problem of determining a surface of minimum area bounded by a given contour, has its corresponding Euler-Ostrogradski equation of the form

$$\frac{\partial^2 f}{\partial x^2}\left[1 + \left(\frac{\partial f}{\partial y}\right)^2\right] - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\left[1 + \left(\frac{\partial f}{\partial x}\right)^2\right] = 0\ .$$

Generalizing to a functional of type

$$J = \iiint \cdots \int I \; f, f_{x_1}, f_{x_2}, \ldots, f_{x_n}, x_1, x_2, \ldots, x_n \; dx_1 dx_2 \cdots dx_n \qquad (11.2.78)$$

whose Euler's equation is:

$$\frac{\partial I}{\partial f} - \sum_{i=1}^{n}\left\{\frac{\partial^2 I}{\partial x_i \partial f_{x_i}} + \frac{\partial^2 I}{\partial f \partial f_{x_i}} + \sum_{j=1}^{n}\left[\frac{\partial I}{\partial f_{x_i}\partial f_{x_j}} \cdot \frac{\partial f_{x_j}}{\partial x_i}\right]\right\} = 0 \qquad (11.2.79)$$

Another type of functional that appears in engineering problems is of the type:

$$J = \iint I \; f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}, x, y \; dxdy \qquad (11.2.80)$$

whose Euler equation is of the type:

$$\frac{\partial I}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial f_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial I}{\partial f_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial I}{\partial f_{yy}}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial I}{\partial f_{xy}}\right) = 0 \qquad (11.2.81)$$

An example for this type of functional is $J = \iint \; f_{xx}^2 + f_{yy}^2 + 2f_{xy}^2 - 2fc \; dxdy$

whose Euler's equation is $f_{xxxx} + 2f_{xxyy} + f_{yyyy} = c$, or if $\nabla^4 f = c$. It $f = w \; x, y$ represents the vertical deflection of an isotropic plate under bending, and $c = p/d$ (p =uniformly distributed loading on the plate $d = Eh^3/12 \; 1 - \nu^2$ and the stiffness of the plate, the $\nabla^4 w = p/d$ the equation represents the equation of the bending of an isotropic plate under vertical load p uniformly distributed on it, If, on the other hand, the previous functional depended on derivatives higher than the second-order one, we would have:

$$J = \iint I \; f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}, \cdots, f_{x^n}, f_{y^n}, f_{x^{m-n} y^m}, x \; dxdy \qquad (11.2.82)$$

Euler's equation is:

$$\sum_{k=0}^{k=n}\sum_{m=0}^{k} -1^{k} \frac{\partial^k}{\partial x^{k-m}\partial y^m}\left(\frac{\partial I}{\partial f_{x(k-m)y(m)}}\right) = 0 \qquad (11.2.83)$$

In a functional type:

$$J = \iint I \quad f,g,f_x,g_x,f_y,g_y,x,y \quad dxdy \qquad (11.2.84)$$

Euler's equation represented by the following system:

$$\frac{\partial I}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial f_y}\right) = 0 \qquad (11.2.85)$$

$$\frac{\partial I}{\partial g} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial g_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial g_v}\right) = 0$$

An example of this type of two-dimensional functional is one that represents the internal energy of a plate in the flat state of tension:

$$F(u,v) = \left(\frac{Eh}{2(1-\nu^2)}\right)\iint_{\Omega}\left\{\begin{matrix} u_v^{\ 2} + u_y^{\ 2} + 2\nu.u_x v_y + \\ +\frac{1}{2}(1-v)\left[u_y^{\ 2} + v_x^{\ 2} + 2\nu.u_y v_x\right]\end{matrix}\right\}dxdy$$

(11.2.86)

where *And* = modulus of elasticity; *h* = thickness of the plate; ν = Poisson's coefficient; *u* and *v* The displacements in the directions *x* and *y* . Euler's equations for the functional above, represent the equilibrium equations for the plate in terms of x,y, and are as follows:

$$\frac{Eh}{1-v^2}\left[u_{xx} + v.v_{xy} + \left(\frac{1-v}{2}\right)(u_{yy} + v_{xy}\right] = 0$$

$$\frac{Eh}{1-v^2}\left[v_{xx} + v.u_{xy} + \left(\frac{1-v}{2}\right)(v_{yy} + u_{xy}\right] = 0$$

2.1.9.2 Exercises

1. Show that the conditions necessary for the stationary character of integrals:

a - $\int_{x1}^{x2} F(x,y,z,\ldots,y',\ldots,z',\ldots)dx$

b - $\int_{\Omega} F(x,y,u,ux,uy)dxdy$

c - $\int_{\Omega} F(x,y,u,ux,uy,vx,vy,\ldots)dxdy$

are respectively

a - $F_{y'} = F_{z'} = 0 \therefore x = x_1 \wedge x = x_2$

b - $F_{u_x}\left(\frac{dy}{ds}\right) - F_{u_y}\left(\frac{dx}{ds}\right) = 0$

c - $\begin{cases} F_{u_x}\left(\frac{dy}{ds}\right) - F_{u_y}\left(\frac{dx}{ds}\right) = 0 \\ F_{v_x}\left(\frac{dy}{ds}\right) - F_{v_y}\left(\frac{dx}{ds}\right) = 0 \end{cases}$

at the contour Ω where s is the length of this contour. Finding the extremes of the functional

a - $J\ y\ x = \int_0^1 y^2 + 2y'^2 + y''^2\ dx$ com $\begin{cases} \grave{}y\ 0 = y\ 1 = 0 \\ y'\ 0 = 1; y'\ 1 = -\sinh\ 1 \end{cases}$

b - $J\ y\ x = \int_0^1 2y'^2 + y''^2\ dx$ com $\begin{cases} y\ 0 = 0; \quad y\ 1 = 1 \\ y'\ 0 = 1; \quad y'\ 1 = \cosh\ 1 \end{cases}$

2.1.10 Quadratic Functionals

Definition 1: It can also be defined as being a quadratic functional, to the $J : V \rightarrow \mathbb{R}$, defined on a *class V* of *functions f x*, if it satisfies the identity (see definition 11 above):

$$J\ u + \tau\eta = J\ u + \tau\delta J\ u;\eta + \frac{1}{2}\tau^2\delta^2 J\ u;\eta \tag{11.2.87}$$

Com $\forall u \in V, \quad \forall \eta \in \tilde{V}, \quad \|\eta\| = 1, \quad \forall \tau \in \mathbb{R}.$

Definition 2: A quadratic functional is a functional in form

$$f\ \mathbf{x} = \frac{1}{2}\mathbf{x}^T H \mathbf{x} - \mathbf{b}^T\mathbf{x} + c, \tag{11.2.88}$$

$\therefore \mathbf{x} \in \mathbb{R}^n, H \in \mathbb{R}^{n \times n}(\text{com } H^T = H),\ b \in \mathbb{R}^n,\ c \in \mathbb{R}$

As previously seen, the gradient and Hessian of the above functional are easily found:

$$\begin{cases} \mathbf{g}\ \mathbf{x} = H\mathbf{x} - \mathbf{b} \\ H\ \mathbf{x} = H \end{cases} \tag{11.2.89}$$

Thus, it can be seen that the Quadratic Functionals have the constant Hessian.

Taking ξ as a stationary point of f, then the gradient cancels out on it and you have

$$\mathbf{g}\ \xi\ = \nabla f\ \xi\ = H\xi - \mathbf{b} = 0 \Rightarrow \xi = H^{-1}\mathbf{b}$$

Making $\mathbf{x} = \mathbf{x} - \xi$ in the quadratic functional has

$$f\ \mathbf{x}\ -\xi\ = \frac{1}{2}\ \mathbf{x}-\xi\ ^{T} H\ \mathbf{x}-\xi\ - \mathbf{b}^{T}\ \mathbf{x}-\xi\ + c,\ \therefore \mathbf{x} \in \mathbb{R}^{n}, \\ = \frac{1}{2}\ \mathbf{x}-\xi\ ^{T} H\ \mathbf{x}-\xi\ + \chi,\quad \therefore \chi = -\mathbf{b}^{T}\ \mathbf{x}-\xi\ + c \tag{11.2.90}$$

]Being H symmetrical and $\lambda_i; \mathbf{v}_i\ {}_{i=1}^{n}$ The set of auto solutions H then

$$\lambda_i; \mathbf{v}_i\ {}_{i=1}^{n} \rightarrow \begin{cases} H\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ i = 1,2,\ldots,n \\ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \\ \mathbf{v}_i \mathbf{v}_j = \delta_{ij},\ i,j = 1,2,\ldots,n \end{cases} \tag{11.2.91}$$

Defining the Diagonal Matrix $\Lambda_{n\times n}$ whose diagonals are the eigenvalues of H and the matrix $V_{n\times n}$ n whose columns are the eigenvectors of H We have:

$$\Lambda = diag\left[\lambda_1, \lambda_2, \ldots, \lambda_n\right]; \quad V = \left[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\right]$$

As defined $\Lambda_{n\times n}$ e $V_{n\times n}$ it is seen that the matrix V is orthogonal, i.e. $V^{-1} = V^{T}$ and $HV = V\Lambda$. Thus, the following expression is reached:

$$f\ \mathbf{x}\ -\xi\ = \frac{1}{2}\ \mathbf{x}-\xi\ ^{T} H\ \mathbf{x}-\xi\ + \chi,\quad \therefore \chi = -\mathbf{b}^{T}\ \mathbf{x}-\xi\ + c$$

Doing in $\mathbf{z} = V^{T}\ \mathbf{x} - \xi$ the previous expression if you have

$$\hat{f}\ z\ \equiv f\ V\mathbf{z} - \xi\ = \frac{1}{2} z^{T} V^{T} H V \mathbf{z} + \chi,\quad \therefore \chi = -\mathbf{b}^{T}\ \mathbf{x}-\xi\ + c \\ = \tfrac{1}{2}\mathbf{z}^{T}\Lambda\mathbf{z} + \chi \\ = \tfrac{1}{2}\sum_{i=1}^{n} \lambda_i z_i^2 + \chi \tag{11.2.92}$$

Looking at the expression (11.2.92), we see that

- If H is SPD (symmetric and positive defined) then its eigenvalues are all positive and the interval $of\ f\ is\left[\chi, \infty\right. = 0$ *and* z ***is a*** global minimizer *of* f.
- If H is negative defined, the range of f where the eigenvalues are located is $\left.-\infty, \chi\right]$, e $\mathbf{z} = 0$ It's a *Global Strong Maximizer*.

- If H has positive and negative eigenvalues, the interval of *f* is , $-\infty, \infty$ in is , $\mathbf{z} = 0$ f stationary, because it is neither maximum nor minimum.

Note that when *H* for SPD (symmetric and positive defined) and when $k > \chi$ the Level surface *Lk* (equation (11.1.47).) It's the ellipsoid:

$$\hat{k} = \sum_{i=1}^{n} \lambda_i z_i^2 \therefore \hat{k} = 2\ k - \chi \qquad (11.2.93)$$

The Dk distortion measure can be evaluated by the so-called spectral condition number of *H*[1], given by:

$$\kappa\ H = \frac{\lambda_n}{\lambda_1} \equiv \frac{\lambda_{max}\ H}{\lambda_{min}\ H} \qquad (11.2.94)$$

Looking at the following figure, which contains graphs for several quadratic shapes, the following can be identified:

FIGURE 3 CHARTS OF QUADRÁTICAS[2] SHAPES

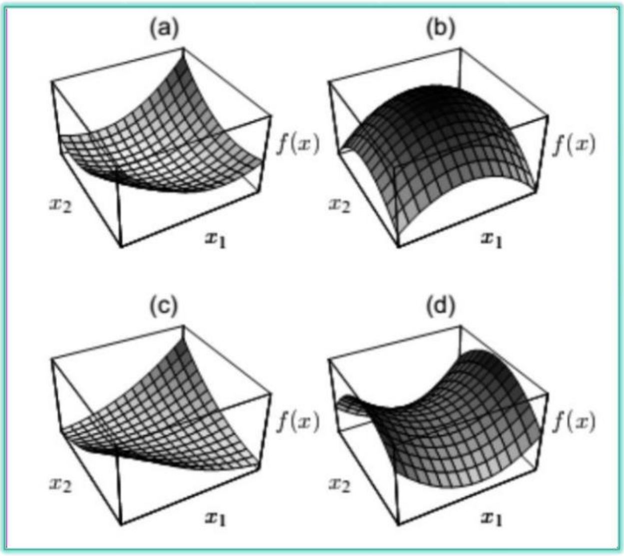


Figure (a) represents the graph from a Quadratic Form to a positive-defined matrix;
Figure (b) is the graph representing a quadratic shape that has a negative-defined matrix.
Figure (c) shows the graph for a singular (and positive-undefined) matrix. The line that runs at the base

[1] Do not confuse the spectral condition number defined by (11.2.94) with the condition number of an array with $\mathbf{A}$ to a matrix standard, which is given by $\kappa\ A = \|A\|\|A^{-1}\|$.
2 Source: [211] Jonathan Richard Shewchuk, An Introducyion to Conjugate Gradient Method without the Agonising Pain, publicado pela School of Computer Science, Carnegie Mellon University, Pittsburgh, PA, em 1994

of the valley is the set of solutions.

Graph (d) is representative of a shape with an undefined matrix. The solution is a saddle stitch.

The properties of Quadratic Functionals are relevant to non-Quadratic Functionals.

quadratics in $C^2 X$. To see this, one must compare Taylor's expansion to

the Quadratic Functionals (*where H is constant*) and the Taylor expansion to Non-Quadratic Functionals:

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \mathbf{g}^T(\mathbf{x})\mathbf{h} + \tfrac{1}{2}\mathbf{h}^T H\mathbf{h} + o(\|\mathbf{h}\|^2) \tag{11.2.95}$$

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \mathbf{g}^T(\mathbf{x})\mathbf{h} + \tfrac{1}{2}\mathbf{h}^T H(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|^2)$$

Thus it can be seen that in the vicinity of ***x*** the arbitrary functional behaves like a quadratic functional. As we have already seen, if you do $\mathbf{h} = \tau\mathbf{y}$ com $\|\mathbf{y}\| = 1$ and remember the directional derivatives, you know that

$$f(\mathbf{x}+\tau\mathbf{y}) = \sum_{k=0}^{m-1}\frac{\tau^k}{k!}f^{(k)}(\mathbf{x};\mathbf{y}) + \frac{\tau^m}{m!}f^{(m)}(\mathbf{x}+\theta\tau\mathbf{y};\mathbf{y})$$

$$\therefore f^{(m)}(\mathbf{x};\mathbf{y}) = \left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + \cdots + y_n\frac{\partial}{\partial x_n}\right)^m f(\mathbf{x})$$

for k= 0,1,2 if you have the identity

$$f(\mathbf{x}+\tau\mathbf{y}) = f(\mathbf{x}) + \tau f^{(1)}(\mathbf{x};\mathbf{y}) + \frac{1}{2}\tau^2 f^{(2)}(\mathbf{x};\mathbf{y})$$

$$\text{com } f^{(1)}(\mathbf{x};\mathbf{y}) = \mathbf{g}^T(\mathbf{x})\mathbf{y}, \quad \text{e } f^{(2)}(\mathbf{x};\mathbf{y}) = \mathbf{y}^T H(\mathbf{x})\mathbf{y}$$

3 DOWNWARD STEP METHOD

One of the most important methods for asymptotic evaluation of certain types of functional problems, including integral problems, is known as the Downward Step Method. This method has its origin in the observation made by in connection with the estimation of an integral coming from the theory of probability, in the form of:

$$In = \int_a^b f(x)[g(x)]^n dx = \int_a^b f(x)e^{n\psi(x)}dx, \quad n \to +\infty \tag{11.3.1}$$

Where $f(x)$ and $g(x)$ are real functions that are continuous and defined in an interval $[a,b]$, which can also be infinite, with $g(x) > 0$ e $\psi(x) = \log g(x)$ Laplace argued that the dominant contribution of this when $n \to \infty$ it would need to come from the vicinity of the point where $g(x)$ (or $\psi(x)$)

would have its maximum value. In the simplest case is the situation Where $\psi\ x$ owning a point maximum $x = \xi \in [a,b]$ so that $\psi'\ \xi\ = 0$, $\psi''\ \xi\ < 0$, e $f\ \xi\ \neq 0$.

More details can be seen in the work of Paris, R. B. (Richard Bruce), entitled "Hadamard Expansions and Hyperasymptotic Evaluation : An Extension of the Method, of Steepest Descents", veja referencia [241]

Here, as the focus is on a matrix treatment of equations from functionals, we will present an iterative method to find a strong local minimizer of a quadratic functional and with Hessian SPD, of the type shown in (11.2.88).

It will be shown that the number of iterations required to ensure that the error under the energy standard (energy standard) is less than a number ε times the initial error is limited by:

$$\frac{1}{2}\kappa\ H\ \ln\ 1/\varepsilon \tag{11.3.2}$$

Where $\kappa\ H$ is the spectral condition number of *H*.

The concept of preconditioning was introduced in chapter 7 of volume 1 of this work, and it will be used here to reduce the above limit.

In general, the methods for determining a minimizer of a functional *f* do not $\mathbb{R}^n$ have the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \tau_k \mathbf{d}_k \tag{11.3.3}$$

Wher $\mathbf{d}_k$ is the search direction and τ_k is chosen in such a way as to minimize, or at least reduce, f over some interval of the line passing through in $\mathbf{x}_k$ the direction $\mathbf{d}_k$.

Thus, there are two associated problems:

1. the choice of $\mathbf{d}_k$, and
2. F Inspection on the line, with $\mathbf{x} = \mathbf{x}_k + \tau_k \mathbf{d}_k$, $\tau \in\ -\infty, \infty$

Definition 1: (downward direction). Let be a functional *f and* let be the vectors **x** and **d** , then there exists a number $\tau_0 > 0$ such that

$$f\ \mathbf{x} + \tau\mathbf{d}\ < f\ \mathbf{x}\ ,\quad 0 < \tau \leq \tau_0 \tag{11.3.4}$$

where **d** is a downward direction to *f* at **x**.

Theorem 1: Let $f \in C^1\ \mathbb{R}^n$ and let g be x (or $\nabla f\ \mathbf{x}$) the gradient of *f* in **x** . If a vector d satisfies the $\mathbf{g}^T\ \mathbf{x}\ \mathbf{d} < 0$, so d is a downward direction to *f* at **x**.

$$f(\mathbf{x}+\mathbf{d}) = f(\mathbf{x}) + \mathbf{g}^T(\mathbf{x})\mathbf{d} + \tfrac{1}{2}\mathbf{d}^T H(\mathbf{x})\mathbf{d} + o(\|\mathbf{d}\|^2)$$

$$\therefore \boldsymbol{\sigma}^T(\mathbf{x})\mathbf{d} = 0 \Rightarrow \text{Stationary point}$$

$$\mathbf{d}^T H(\mathbf{x})\mathbf{d} < 0 \Rightarrow f(\mathbf{x}+\mathbf{d}) = f(\mathbf{x}) - \tfrac{1}{2}\mathbf{d}^T H(\mathbf{x})\mathbf{d}$$

Thus we see that an increase in the direction d there is a decrease in the value of the function, so **d** is a downward direction.

Theorem 3: Let $f \in C^1(\mathbb{R}^n)$ So of all the search directions **d** at the same point $\mathbf{x}$ **, the direction to** which f *descends most rapidly is in a vicinity of* x **in which** $\mathbf{d} = -\mathbf{g}^T(\mathbf{x})$.

Proof:

Since you want to minimize the directional derivative of f in **x** in all directions, you have $f^1(\mathbf{x};\mathbf{y}) = \mathbf{g}^T(\mathbf{x})\mathbf{y} \therefore \|\mathbf{y}\| = 1$, this is the same as minimizing $\mathbf{g}^T(\mathbf{x})\mathbf{y}$ for all **y** so that you have $\|\mathbf{y}\| = 1$

That way you have

$$\left|\mathbf{g}^T(\mathbf{x})\mathbf{y}\right| \leq \left\|\mathbf{g}^T(\mathbf{x})\right\| . \|\mathbf{y}\| = \left\|\mathbf{g}^T(\mathbf{x})\right\| \Rightarrow \mathbf{y}_{\min} = -\mathbf{g}(\mathbf{x}) / \|\mathbf{g}(\mathbf{x})\| \qquad (11.3.5)$$

Analyzing the problem of determining τ_k data $\mathbf{x}_k$ and $\mathbf{d}_k$ (review theorem 5 above) where $f(\mathbf{x})$ is minimized over the line $\mathbf{x} = x_k + \tau_k \mathbf{d}_k, \quad \tau_k \in (-\infty, \infty)$.

Any procedure to determine is called a "search line" or "search line".

For generic functionals, the lines of research are, in general, quite complicated and involve some iterative process. In the case of quadratic functionals with the Hessian SPD, the process is simplified and a simple formula can be deduced for . If τ_k

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T H \mathbf{x} - b^T \mathbf{x} + c \rightarrow$$

$$f(\mathbf{x} + \tau \mathbf{d}) = \tfrac{1}{2}\tau^2 d^T H(\mathbf{x})\mathbf{d} + \tau d^T g(\mathbf{x}) + \chi \therefore \chi \quad \tau \qquad \text{Independence of (11.3.6)}$$

If H is SPD $\mathbf{d} \neq 0$ then $\mathbf{d}^T H \mathbf{d} > 0$ e $f(\mathbf{x} + \tau\mathbf{d})$ is a parabola in the variable and is τ open upwards so that $f(\mathbf{x} + \tau\mathbf{d})$ is uniquely minimized by (regardless of the choice of $\mathbf{d}_k$:

$$\tau = -\mathbf{d}^T \mathbf{g}(\mathbf{x}) / \mathbf{d}^T H \mathbf{d} \quad \Rightarrow \quad \tau_k = -\mathbf{d}_k^T \mathbf{g}(\mathbf{x}_k) / \mathbf{d}_k^T H \mathbf{d}_k \qquad (11.3.7)$$

3.1 DESCENDING STEP METHOD ALGORITHM

Based on Theorem 3 above, our natural choice for the downward direction **d** is $\mathbf{d} = -\mathbf{g}(\mathbf{x}_k)$.

Thus, the iterative process will take place by:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \tau_k \mathbf{g}(\mathbf{x}_k) \qquad (11.3.8)$$

This relationship, with some strategy of choosing the "line of research", defines the method of the descending step, to minimize a generic functional. For a quadratic functional, it was seen that the way to determine is: τ_k

$$\tau_k = -\mathbf{d}_k^T \mathbf{g}\ \mathbf{x}_k \ / \mathbf{d}_k^T H \mathbf{d}_k \tag{11.3.9}$$

Based on what has been presented here, the descending step method for a quadratic functional follows the following algorithm:

$$\left.\begin{aligned} \mathbf{g}_k &= H\mathbf{x}_k - \mathbf{b} \\ \tau_k &= \mathbf{g}_k^T \mathbf{g}_k \ / \ \mathbf{g}_k^T H \mathbf{g}_k \\ \mathbf{x}_{k+!} &= \mathbf{x}_k - \tau_k \mathbf{g}_k \end{aligned}\right\} \therefore k = 0,1,\ldots; \wedge \mathbf{g}_k = \mathbf{g}\ \ \mathbf{x}_k \tag{11.3.10}$$

In the (k+1)-th iteration, the line connecting $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ is tangent to the point $\mathbf{x}_{k+1}$ belonging to the ellipsoidal level surface given by $\mathbf{x} \in \mathbb{R}^n; f\ \ \mathbf{x}\ \ = f\ \ \mathbf{x}_{k+1}$

Taking the interative process $\mathbf{x}_{k+1} = \mathbf{x}_k - \tau_k \mathbf{g}\ \ \mathbf{x}_k$ by pre-multiplying the expression by H and subtracting b on both sides, we get:

$$\underbrace{H\mathbf{x}_{k+1} - \mathbf{b}}_{\mathbf{g}_{k+1}} = \underbrace{H\mathbf{x}_k - \mathbf{b}}_{\mathbf{g}_k} - \tau_k H \mathbf{g}\ \ \mathbf{x}_k \ \ \Rightarrow \mathbf{g}_{k+1} = \mathbf{g}_k - \tau_k H \mathbf{g}_k \tag{11.3.11}$$

Redefining the previous algorithm has:

$$\left.\begin{aligned} \tau_k &= \mathbf{g}_k^T \mathbf{g}_k \ / \ \mathbf{g}_k^T H \mathbf{g}_k \\ \mathbf{x}_{k+1} &= \mathbf{x}_k - \tau_k \mathbf{g}_k \\ \mathbf{g}_{k+1} &= \mathbf{g}_k - \tau_k H \mathbf{g}_k \end{aligned}\right\} \therefore k = 0,1,\ldots; \wedge \mathbf{g}_0 = H\mathbf{x}_0 - \mathbf{b} \tag{11.3.12}$$

3.2 CONVERGENCE ANALYSIS

3.2.1 Choosing the Step Size for the Search Line

For first-order approximations, each step decreases the value of *f* by approximately $\tau_k \mathbf{g}_k = \tau_k \left\| \nabla f\ \ x_k \ \right\|^2$ τ_k tag. If it's too small, the algorithm will converge too slowly. On the other hand, if the step size is not conveniently small, the algorithm may fail to reduce *f*. A suitable way is to adopt a step size so that it is sufficient for the reduction of *f*, and the algorithm should progress as quickly as possible. This procedure is known as *a search line* and is employed in many other multivariate optimization algorithms.

3.2.2 Analysis

It is necessary to measure the quantitative convergence of every iterative process. One can use the notion of how close $\mathbf{x}$ **is to** ξ using the Euclidean norm $\|\mathbf{x}-\xi\|_2$ which is the error defined by the classical distance between x **and** ξ.

Since these are functional and these represent, in general, when they come from a physical problem, an energy system, it is more appropriate to measure this error by the quantity $f(\mathbf{x}) - f(\xi)$ Thus, it is more important to make $f(\mathbf{x}) - f(\xi)$ small than make small $\|\mathbf{x}-\xi\|_2$. It is then necessary to define a standard for measuring energy: the energy standard.

Definition 2: The *energy internal product* and the *energy standard* corresponding to a defined positive H matrix are respectively

$$\langle \mathbf{x},\mathbf{y}\rangle_H = \mathbf{x}^T H \mathbf{y} \tag{11.3.13}$$

$$\|\mathbf{x}\|_H = \langle \mathbf{x},\mathbf{x}\rangle_H^{\frac{1}{2}} = \left(\mathbf{x}^T H \mathbf{x}\right)_H^{\frac{1}{2}} \tag{11.3.14}$$

Note that these definitions obey the axioms of internal product and norm respectively.

Note that when H is the identity matrix, the energy dot product and the energy norm reduce to the Euclidean dot product and the Euclidean norm respectively. Designating the square of the energy standard by $E(\mathbf{x})$, we have

$$\mathbf{E}(\mathbf{x}) = \left(\|\mathbf{x}\|_H\right)^2 = \left(\left(\mathbf{x}^T H \mathbf{x}\right)_H^{\frac{1}{2}}\right)^2 = \mathbf{x}^T H \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{11.3.15}$$

Taking the first expression of (11.3.6) and transforming it, we find

$$\left.\begin{aligned} f(\mathbf{x}) &= \tfrac{1}{2}\mathbf{x}^T H \mathbf{x} - b^T \mathbf{x} + c \\ f(\xi) &= \tfrac{1}{2}\xi^T H \xi - b^T \xi + c \end{aligned}\right\} \rightarrow f(\mathbf{x}) - f(\xi) = \tfrac{1}{2}\mathbf{x}^T H \mathbf{x} - b^T \mathbf{x} + \not{c} - \tfrac{1}{2}\xi^T H \xi + b^T \xi - \not{c}$$

$$f(\mathbf{x}) - f(\xi) = +\tfrac{1}{2}\mathbf{x}^T H \mathbf{x} - \tfrac{1}{2}\xi^T H \xi + b^T \xi - b^T \mathbf{x}$$

$$f(\mathbf{x}) - f(\xi) = +\tfrac{1}{2}(\mathbf{x}-\xi)^T H (\mathbf{x}-\xi) - b^T(\mathbf{x}-\xi)$$

$$\text{mas } E(\mathbf{x}) = \mathbf{x}^T H \mathbf{x} \quad e \quad b^T(\mathbf{x}-\xi) \rightarrow 0 \quad \text{logo}$$

$$f(\mathbf{x}) - f(\xi) = +\tfrac{1}{2}E(\mathbf{x}-\xi) \Rightarrow \begin{cases} E(\mathbf{x}-\xi) = 2\left(f(\mathbf{x}) - f(\xi)\right) \\ \|\mathbf{x}-\xi\|_H = \sqrt{2\left(f(\mathbf{x}) - f(\xi)\right)} \end{cases}$$

Soon

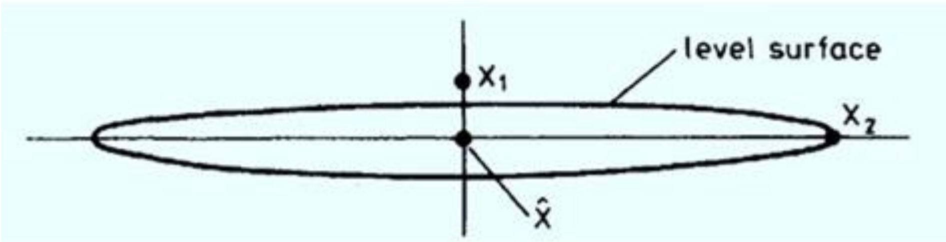
$$\mathbf{E}\ \ \mathbf{x}-\xi\ \ =2\ f\ \ \mathbf{x}\ \ -f\ \ \xi \qquad (11.3.16)$$

$\rightarrow$

$$\|\mathbf{x}-\xi\|_H=\sqrt{2\ f\ \ \mathbf{x}\ \ -f\ \ \xi} \qquad (11.3.17)$$

are the relationships between the energetic norm **and E x** and the energy or functional $f\ \mathbf{x}$. Note that $E\ \ \mathbf{x}-\xi$ e $\|\mathbf{x}-\xi\|_H$ are constant on every surface level of F.

Why use energy norms instead of using the Euclidean norm? As can be seen, the level surfaces are ellipsoidal, and when the eccentricity increases (making the eccentricity larger) the values of these quoted norms become more and more pronounced: $\|\mathbf{x}_2-\xi\|_H<\|\mathbf{x}_1-\xi\|_H$ while as $\|\mathbf{x}_1-\xi\|<\|\mathbf{x}_2-\xi\|$ can be seen in the following figure:



To avoid this, the preconditioning technique can be used, where H is exchanged for a matrix whose level surfaces are significantly less eccentric than those of H.

So, if you are interested in knowing what is the convergence rate of MPD – Descending Step Method or Gradient Method.

Theorem 4: Since H is a definite symmetric and positive matrix, then the descending step method (or gradient method) is convergent to any choice of the initial Datum $\mathbf{x}_0$ and further, having

$$\|\mathbf{x}_k-\xi\|_H\leq\left(\frac{\kappa\ \ H\ \ -1}{\kappa\ \ H\ \ +1}\right)^k\|\mathbf{x}_0-\xi\|_H \qquad (11.3.18)$$

Where $\kappa\ \ H\ \ =\lambda_n/\lambda_1\geq 1$is the spectral condition number of the matrix H and $\|\cdot\|_H$ [and the energy norm defined higher up. If in addition, if $p\ \ \varepsilon$ is defined for any , $\varepsilon>0$ to be the smallest integer k such that

$$\|\mathbf{x}_k-\xi\|_H\leq\varepsilon\|\mathbf{x}_0-\xi\|_H\qquad\forall\mathbf{x}_0\in\mathbb{R}^n \qquad (11.3.19)$$

So the smallest integer k that meets (11.3.18) is given by

$$\text{menor } k = p \quad \varepsilon \quad \leq \tfrac{1}{2}\kappa \quad H \quad \ln\left(\frac{1}{\varepsilon}\right) + 1. \tag{11.3.20}$$

3.3 PRECONDITIONING

What is preconditioning?

To know this mechanism, we suggest topic 7.7 of the book by prof. Henrique Mariano "Analysis and Numerical Methods in Engineering", which will be remembered here: "To reduce the spectral conditional number (ratio between the highest eigenvalue and the lowest eigenvalue) of a matrix, $\mathbf{A}$, $\kappa \quad \mathbf{A}$ and thus improve the performance of iterative algorithms, one usually swaps the original system for another system that has the same solution. It is known that when the conditional number of **A** is large, the matrix tends to be badly conditioned, and it is necessary to condition it."

Hypothetically, a **symmetric and positive-defined** matrix A (SPD) is and an **M** preconditioner is available.

Definition 3: A *preconditioned* **M** is a matrix that approximates a matrix **A** in some sense, for example $\mathbf{M}^{-1}\mathbf{A} = I$.

3.3.1 Domestic Product and Energy Standard

Definition 4: The *energy dot product (or H - dot product)* and the energy **norm (or** H - *norm), corresponding to any defined positive matrix, are respectively given by, similarly to the expressions (11.3.13) and (11.3.14) for the Hessian matrix and which coincide when*

$\mathbf{H} = \mathbf{M}$:

$$\mathbf{x}, \mathbf{y} \quad _H = \mathbf{x}^T\mathbf{M}\mathbf{y} \tag{11.3.21}$$

$$\|\mathbf{x}\|_H = \quad \mathbf{x}, \mathbf{x} \quad ^{\frac{1}{2}} = \quad \mathbf{x}^T\mathbf{M}\mathbf{x} \quad ^{\frac{1}{2}}$$

and satisfy all valid properties for Euclidean inner product and norm

Euclidean. Note that when , $\mathbf{M} = I$, *the H-dot product coincides with the*

Euclidean and the H-norm with the Euclidean norm (as explained above).

Assuming that the matrix M is **symmetric and positive-defined, then, from a practical point of view, the only requirement** for M to be a preconditioner is that it induces an easy solution for a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

A preconditioned system takes the following form (see eq. (7.7.1) of[4]:

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b} \tag{11.3.22}$$

Next, the concept of preconditioning will be applied to a quadratic functional.

Let M **be** a definite positive symmetric matrix factored into the form $\mathbf{M} = \mathbf{L}\mathbf{L}^T$ (lower Cholesky decomposition) and is the quadratic functional, as defined in (11.2.88):

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \chi, \quad \mathbf{x} \in \mathbb{R}^n$$

where **H** is positive defined. Defining a functional second $\tilde{f}(\mathbf{y})$ by the transformation ,$\mathbf{y} = \mathbf{L}^T\mathbf{x}$ we have

$$\tilde{f}(\mathbf{y}) = f(\mathbf{L}^{-T}\mathbf{y}) = \frac{1}{2}\mathbf{y}^T\tilde{\mathbf{H}}\mathbf{y} - \tilde{\mathbf{b}}^T\mathbf{y} + \tilde{\chi} \tag{11.3.23}$$

where

$$\tilde{\mathbf{H}} = \mathbf{L}^{-1}\mathbf{H}\mathbf{L}^{-T}, \quad \tilde{\mathbf{b}} = \mathbf{E}^{-1}\mathbf{b}, \quad \tilde{\chi} = \chi \tag{11.3.24}$$

Since, by definition, **H** is positive definite, it $\tilde{\mathbf{H}} = \mathbf{L}^{-1}\mathbf{H}\mathbf{L}^{-T}$ is a similarity transformation, so it is also $\tilde{\mathbf{H}}$ symmetric and positive definite

The Similarity Transformation

$$\mathbf{L}^{-T}\tilde{\mathbf{H}}\mathbf{L}^{T} = \mathbf{L}^{-T}\mathbf{L}^{-1}\mathbf{H}\underbrace{\mathbf{L}^{-T}\mathbf{L}^{T}}_{=I} = \underbrace{\mathbf{L}^{-T}\mathbf{L}^{-1}}_{\mathbf{M}^{-1}}\mathbf{H} = \mathbf{M}^{-1}\mathbf{H} \tag{11.3.25}$$

It reveals that both $\tilde{\mathbf{H}}$ and $\mathbf{M}^{-1}\mathbf{H}$ have the same eigenvalues, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ so that the spectral condition $\kappa(\tilde{\mathbf{H}}) = \lambda_n/\lambda_1$ number is completely determined by M **and** H since it $\tilde{\mathbf{H}}$ depends on the factorization of M **.**

Applying the descending step method to the problem (11.3.23), it is seen that it is convenient to use the direct calculation of the gradients and then the iterative process is

$$\tilde{\mathbf{g}}_k = \tilde{\mathbf{H}}\mathbf{y}_k - \tilde{\mathbf{b}} \tag{11.3.26}$$

$$\tilde{\tau}_k = \tilde{\mathbf{g}}_k^T\tilde{\mathbf{g}}_k / \tilde{\mathbf{g}}_k^T\tilde{\mathbf{H}}\tilde{\mathbf{g}}_K \tag{11.3.27}$$

$$\mathbf{y}_{k+1} = \mathbf{y}_k - \tilde{\tau}_k\tilde{\mathbf{g}}_k \tag{11.3.28}$$

For $k = 0,1,\ldots$ and being the arbitrarily $\mathbf{y}_0$ chosen starting point one has

$$\lim_{k\to\infty} \mathbf{y}^k = \psi \equiv \tilde{\mathbf{H}}^{-1}\tilde{\mathbf{b}} \tag{11.3.29}$$

The convergence rate of (11.3.29) depends on $\kappa(\tilde{\mathbf{H}})$.

Making $\mathbf{x}^k = \mathbf{L}^{-T}\mathbf{y}^k$ e $\mathbf{g}^k = H\mathbf{x}^k - \mathbf{b}$ for $k = 0,1,2,\ldots$

4 BOUNDARY VALUE PROBLEMS: VARIATIONAL FORMULATION

In engineering, the vast majority of problems are formulated as boundary value problems (PVC), which can be expressed in the determination of a function that satisfies some differential equation in a definition region Ω and that must satisfy specific conditions in the boundary Γ of the region Ω. In general, the vast majority of these problems are related to a solution that minimizes a functional J f defined to functions f belonging to a set of functions V.

This minimization (or maximization) requires the functional $J f$ to be stationary. Thus, the task of solving a PVC is equivalent to finding a function, a function $f \in V$ that makes $J f$ stationary. This is the variational formulation of a PVC.