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Approximation Methods in Variational Problems

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ABSTRACT
This research material suggests the exploration of approaches to deal with variational problems through approximation techniques. In mathematical contexts, variational problems involve optimization of functions, and approximation methods seek to find approximate solutions to these problems. These

approaches can be essential in situations where finding an exact solution is challenging or impractical, allowing for the effective analysis and resolution of complex issues through approximation techniques.

Keywords: Approximation methods, infinite series, MATLAB, boundary conditions.

1 GENERAL CONSIDERATIONS

Here we will introduce some concepts from Lebesgue's theory of integration[1] and Sobolev's spaces. The reader is recommended to study *chapter 2 of volume 1* of this work in order to recall basic concepts of linear spaces.

Definition 1: It is called *domain* Ω, in Lebesgue's sense, to a subset (open or closed) of $\mathbf{R}^n$ with non-empty interior.

Definition 2: This is called _the Lebesgue measurable integral of functions f over a given domain Ω a

$$\cdot \int_{\Omega} f x \, dx \quad .$$

Definition 3: Define *norm* of Lebesgue, to the norm (which is a function of $\|*\|$ about a normed Euclidean space whose values are non-negative – *See item 2.9 p.85 in volume 1 of this work*) established by the expression

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f(x)|^p \, dx \right)^{1/p}$$

Definition 4: The Lebesgue space is a Banach[2] space defined by

$$Lp(\Omega) := \{ f : \|f\|_{Lp(\Omega)} < \infty \} \tag{1.1.1}$$

Definition 5: It is said that two functions f and g are equal if they differ from each other by one subset of points of measure zero, i.e.,

$$\|f - g\|_{Lp(\Omega)} \equiv 0 \tag{1.1.2}$$

[1] Henri Leon Lebesgue (June 28, 1875 – July 26, 1941) was a French mathematician.
[2] Stefan Banach (30/mar/1892 – 31/ago/1945), matemático polonês.

We observe that, in a Lebesgue space, because it is a Banach space, Minkowski's[3] inequalities remain valid:

$$\|f+g\|_{Lp\,\Omega} \leq \|f\|_{Lp\,\Omega} + \|g\|_{Lp\,\Omega} \quad \therefore 1 \leq p < \infty; f,g \in Lp\ \Omega\ ;$$

Holder[4]:

$$\|fg\|_{L1\,\Omega} \leq \|f\|_{Lp\,\Omega}\,\|g\|_{Lq\,\Omega} \quad \therefore 1 \leq p,q < \infty; 1 = \frac{1}{p} + \frac{1}{f\,q};\ \in Lp\ \Omega\ , g \in Lq\ \Omega$$

Schwarz[5]:

$$\int |fg|\,d\Omega \leq \|f\|_2^2 \cdot \|g\|_2^2 \therefore \|\cdot\|_2 = \text{2-norma.}$$

It should be noted that the key to the processes for the determination of a numerical solution for differential equations will be the ability to develop precise functions for the approximation methods employed in the definition space of the studied PDD.

To get an overview, whether it's the approximation of a given function in some region $\Omega \subset \mathbb{R}$ whose contour is bounded by a curve . In the solution of EDPs, certain conditions are usually prescribed in its outline; a function is then needed that it satisfies the prescribed boundary conditions, $\psi|_{\Gamma} = \phi|_{\Gamma}$ let then be a set of functions $\varphi_i, i = 1, 2, \ldots$ introduced in such a way that $\forall \varphi|_{\Gamma} = 0$, so that at all the interior points of Ω can be approximated by ϕ

$$\phi \approx \psi + \sum_{i=1}^{N} w_i \varphi_i \tag{1.1.3}$$

The computation of the coefficients a $_i$ is what will make the constructed approximation good or unsatisfactory.

2 RAYLEIGH-RITZ METHOD

The idea of this method is that when finding the extreme of a functional J f is to consider instead of the space of the permissible functions, only those functions that can be represented as linear combinations of the coordinate functions that form a basis of a subspace of the permissible functions (also called basis functions or even permissible functions) are considered:

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[3] Hermann Minkowski (June 22, 1864 – January 12, 1909) was a German mathematician.
[4] Otto Ludwig Holder (Dec 22, 1859 – Aug 29, 1937) was a German mathematician.
[5] Karl Hernann Amandus Schwarz (January 25, 1843 – November 30, 1921), German mathematician;

$$fn\,x = \sum_{j=1}^{n} aj\,\varphi_j\,x \quad (2.1.1)$$

where *aj* are constant.

Definition 1: The system or set of functions φ_j, it's called coordinated functions.

A set is φ_j made up of functions φ_j that are linearly independent and that constitute a complete system of functions in the given space, each of which satisfies exactly the essential boundary conditions, for example:

$$\varphi_j\,a = \varphi_j\,b = 0, \forall = 1,2,\ldots,n \quad (2.1.2)$$

Generally speaking, when you ask that the functions $f\,x$ are admissible, it is necessary to the coordinate functions $\varphi_j\,x$ certain conditions such as limitations on derivability and on the verification of boundary conditions. In this way, the functional J f becomes a function of the aj arguments, i.e.,

$$J\,f_n = \Phi\;a1,a2,\ldots,an \quad (2.1.3)$$

You find the *aj values* that provide extremes to the function by solving the following system:

$$\left(\frac{\partial\Phi}{\partial aj}\right) = 0, \forall = 1,2,\ldots,n \quad (2.1.4)$$

which, as a rule, is non-linear.

The sequence $fn\,x$ thus found converges to the minimum of J f , i.e.,

$$\lim_{n\to\infty} J\,f_n = \min J\;f \quad (2.1.5)$$

However, it cannot be concluded from the previous expression $\lim_{n\to\infty} f_n\;x = f\;x$. The minimizing sequence may not converge to the function that performs the extreme in the class of permissible functions.

Conditions can be indicated that guarantee the existence of the absolute minimum of the functional and that it is achieved in the functions $fn\ x$. In the case of the functional, for example,

$$J(f) = \int_a^b I(f, fx, x)dx \quad with \begin{cases} f(a) = \\ go \quad = a \end{cases} \qquad (2.1.6)$$

These conditions are:

- The function is continuous with respect to the set of its arguments for any $FX \wedge x, f \in$ D , where D is the problem domain;
- Exist Constants $\alpha > 0; p > 1; \beta$ such that $I\ , fx, x \geq |f_\alpha|^p + \beta$
- The I f , fx ,x has continuous partial derivative $\frac{\partial I}{\partial_{Fx}}$, and this is a function descending *whatever x, f* ∈ *D.*

If by this method an absolute minimum of the functional is determined, the approximate value of this minimum is excessive, since the minimum of the functional for arbitrary permissible functions is not greater than the minimum of the latter for a part of the class of permissible functions.

Examples

Find the approximate solution of the problem over the minimum of the functional

$$J(and) \int_0^1 (and'^2 \quad y2 \quad 2xy)Dx$$

Comy 0 = y 1 = 0. Compare the approximate solution with the exact solution.

Solution:

Rayleigh-Ritz Method

Be $\varphi_k\ x = 1 - x\ x^k, k = 1,2,\ldots,n$ the functions coordinated. It can be seen that, $\varphi_k\ x$ by definition,

satisfies the essential boundary conditions and are LI. In addition, they form a complete system in space C1[0,1].

For k= we have $y_1 = a_1\ _1 = a_1\ 1 - x\ x = a_1\ x - x^2$ taking the approximate value of y, y1, in the functional, we have:

$$J\ y_1 = \int_0^1 a_1^2\ 1\quad 2x^{-}\quad a_1^2\ x\quad x^{2\ 2}\ .\ 2a_1\ x\quad x^2\quad dx$$

$$J\ y_1 = \int_0^1 a_1^2\ 1\quad 4x + 3x^2 + 2x^3 - x^4 + 2a_1\ x - x^2\quad dx$$

$$J\ y_1 = \left(\frac{3}{10}\right) a_1^2 \quad \left(\frac{1}{6}\right) a_1 \qquad a_1$$

Solving

$$(\ a_1\ )\qquad 0 \quad \frac{3}{5} a_1 + \frac{1}{6} = 0 \Rightarrow a_1 = -\frac{5}{18}$$

Soon

$$y_1 = -\frac{5}{18}\ x - x^2$$

For k = 2 (a more precise solution than the previous one).

$$y_2 = a_1\ _1 + a_2\ _2 = a_1\ 1 - x\ x + a_2\ 1 - x\ x^2 = a_1\ x - x^2 + a_2\ x^2 - x^3$$

Taking the approximate value of y , y2 , in the functional, and doing the operations, we have:

$$J\ y_2 = \left(\frac{23}{10}\right) a_1^2 \quad \left(\frac{1}{15}\right) a_2^2 \quad \left(\frac{41}{30}\right) a_1 a_2 \quad \left(\frac{1}{\ }\right) a_1 \quad \left(\frac{1}{10}\right) a_2$$

By deriving partially with respect to the parameters and equaling to zero, we obtain:

$$\left(\frac{\partial \Phi}{\partial a_1}\right) = \left(\frac{23}{5}\right) a_1 + \left(\frac{41}{30}\right) a_2 + \left(\frac{1}{6}\right) = 0$$
$$\left(\frac{\partial \Phi}{\partial a_2}\right) = -\left(\frac{2}{15}\right) a_2 - \left(\frac{41}{30}\right) a_1 - \left(\frac{1}{10}\right) = 0$$

Solving the above system is:

$$a_1 = \left(\frac{10615}{101393}\right)$$
$$a_2 = -\left(\frac{799}{2473}\right)$$

soon

$$y_2 = \left(\frac{10615}{101393}\right) 1 - x\ x - \left(\frac{799}{2473}\right) 1 - x\ x^2$$

Exact Solution:

$$J\ y = \int^{I} y',y,x\ dx$$

has the following Euler's equation:

$$\frac{\partial I}{\partial y} - \frac{d}{dx}\left(\frac{\partial I}{\partial y'}\right) = 0$$

How $\partial I/\partial y = -2y - 2x$, $\partial I/\partial y' = 2y'$ e $\partial\ 2y'\ /\partial x = 2y''$

then, forming Euler's equation, we have:

$$-2\ y + x\ - 2y'' = 0$$

That is

$$y'' + y + x = 0$$

which is a second-order homogeneous differential equation, the solution of which is: Let y = x be a particular solution; the general solution of the representative differential equation of Euler's equation is:

$$y = \alpha \cos\ x\ + \beta \sin\ x\ + y \therefore y = -x$$
$$y = \alpha \cos\ x\ + \beta \sin\ x\ - x$$

Using boundary conditions, you get:

$$y\ 0\ = 0 \Rightarrow \alpha \cos\ 0\ + \beta \sin\ 0\ - x = 0 \Rightarrow \alpha = x$$
$$y\ 1\ = 0 \Rightarrow \alpha \cos\ 1\ + \beta \sin\ 1\ - x = x\ \cos\ 1\ - 1\ + \beta \sin\ 1\ = 0$$
$$\Rightarrow \beta = -\frac{x\ \cos\ 1\ - 1}{\sin\ 1} = \frac{x\ 1 - \cos\ 1}{\sin\ 1}$$
$$\beta = \frac{x\ 1 - 0,84147}{0,54030} = 0,546x$$

So the general solution is:

$$y = x\cos x + 0,546x\sin x - x$$
$$y = x(\cos x + 0,546\sin x - 1)$$

Shown below is a comparative study between the y1.y2 and y functions (developed in the $[0,1]$ The values of the three functions are approximate and all satisfy the given boundary conditions.

$$x := 0, 0.05 .. 1$$

$$y1(x) := -\frac{5}{18}\cdot\left(x - x^2\right)$$

$$y2(x) := \frac{10615}{101393}\cdot\left(x - x^2\right) - \frac{799}{2473}\cdot\left(x^2 - x^3\right)$$

$$y(x) := x\cdot(\cos(x) + 0.546\cdot\sin(x) - 1)$$

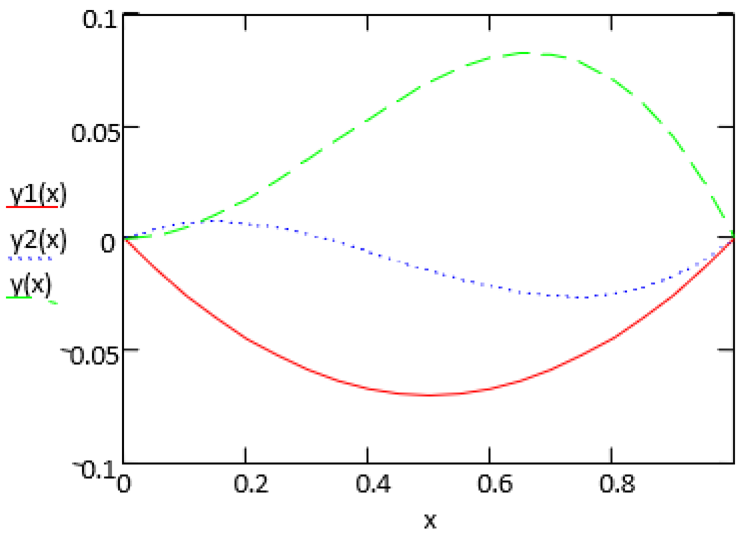


FIGURE 12-1: AUTHOR

Apply the Rayleigh-Ritz Method to the functional

$$J[u] = \int_a^b u u'' dx, \quad \therefore u(a) = \alpha, u(b) = \beta, u \in C^2[a,b] \tag{2.1.7}$$

Solution:

Be choose , where: $u = u_0 + \sum_{i=1}^{n} a_i u_i$.

$$u_0(a) = \alpha, u_0(b) = \beta, \quad u_i(a) = u_i(b) = 0, \forall i \geq 1 \wedge u_i \in C^2[a,b]$$

Which meets the necessary subsidiary conditions. Thus, by replacing itself in the functional, it is possible to

$$J[u] = \int_a^b \left(u_0 u_0'' + \sum_{i=1}^{n} a_i \; u_0 u_i'' + u_i u_0'' \; + \sum_{i,j=1}^{n} a_i a_j u_i u_j'' \right) dx$$

Defining if $= \int_a^b u_i u_j'' dx$, $c_i = -\frac{1}{2}\int \; u_0 u_i'' + u_i u_0'' \; dx$ it has to be functional can be rewritten as follows:

$$J \;\; u \;\; = L_{00} - 2\sum_{i=1}^{n} a_i c_i + \sum_{i,j=1}^{n} L_{ij} a_i a_j$$

The extreme condition is that $\frac{\partial J}{\partial \alpha_i}$ 0 , so $\frac{\partial J}{\partial \alpha_i} = 0 = -2c_i \; + 2\sum_{j=1}^{n} L_{ij} a_j , \quad i = 1,2,...$

In matrix form, you have

$$\begin{bmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ddots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ \vdots \\ c_n \end{Bmatrix}$$

So the solution, which is to find the coefficients of the basis functions, can be reduced in the solution of a system of linear equations:

$$L\alpha = c \quad \Rightarrow \quad \alpha = L^{-1}c$$

Exercises

1 Finding an approximate solution of the nonlinear differential equation $y'' - \frac{3}{2}y^2 \;\; = 0$

Obeying the conditions $y \;\; 0 \;\; = 4; y \;\; 1 \;\; = 1.$

2 Find the minimizing functions of the following functions and compare them with the results of the exact solutions:

a - $J \;\; y \;\; = \int_0^1 \; y'^2 + 2y \; dx \; \therefore y \;\; 0 \;\; = y \;\; 1 \;\; = 0$

b - $J \;\; y \;\; = \int_0^2 \; 2xy + y^2 \; + y'^2 \;\; dx \; \therefore y \;\; 0 \;\; = y \;\; 2 \;\; = 0$

3 WEIGHTED RESIDUALS METHOD (MRP)

3.1 INTRODUCTION

They are approximation methods used to solve differential equations. These methods have several procedures, among which the Method of Moments, the Galerkin Method, the Method of Placement, the Method of Subdomain and the Method of Least Squares stand out.

3.2 BASIC CONCEPTS

Let L be a differential operator of any process, which applied to a function u , produces another function p , in some domain Ω (space where the operator is defined):

$$\mathrm{L}_{\Omega}\ u\ = p \tag{3.1.1}$$

Let be a problem represented by a set of homogeneous equations valid within Ω :

$$\mathrm{L}_{\Omega}\ u\ = 0 \tag{3.1.2}$$

and let the definition of scalar or internal product $c\mathrm{L}_{\Omega}\ u$ with another function v given by:

$$\mathrm{L}_{\Omega}\ u\ ,v\ = \int_{\Omega} \mathrm{L}_{\Omega}\ u\ \cdot v d\Omega \tag{3.1.3}$$

It can be seen that by integrating the above expression in parts, we can successively eliminate the derivatives of u . From the linear analysis, it is known that such a procedure leads to a transposed form of the dot product - with the adjunct operator of L_{Ω} - associated with the terms containing information about the boundary conditions:

$$\int_{\Omega} \mathcal{L}(u).v.d\Omega = \int_{\Omega} u.\mathrm{L}^{*}(v).d\Omega + \int_{\Gamma} \left[\mathbf{G}(v)\mathbf{S}(u) - \mathbf{G}(u)\mathbf{S}^{*}(v)\right].d\Gamma \tag{3.1.4}$$

Where Γ is the contour surface of; **G**,**S** are differential operators due to the integration by parts and L^{*} is the deputy operator of L_{Ω} .

By definition, G v *contains the terms* of v *resulting from the first phase of integration by the part and* S u *the corresponding terms in* u.

If $\mathrm{L} = \mathrm{L}^{*}$ is said to be adjunct auto, and in this case we also have S = S*.

The above piecemeal integration also leads to two categories of boundary conditions:

- The set **G** v prescribed is called Essential Boundary Condition; and
- the prescribed S u set is termed Natural Contour Condition.

The boundary conditions of the essential type need to be known at some points in order to enable the uniqueness of the solution. Let $\Gamma_1 \Gamma_2$ complementary portions of the total surface area be Γ $(\Gamma_1 + \Gamma_2 = \Gamma)$

), then, for a adjoint auto operator L, we have:

$G\,v$, prescribed on Γ_1

$S\,u$, prescribed over Γ_2

It should be remembered that every auto-adjoint operator is positive-defined if:

$$\int_\Omega \mathrm{L}\; u \;\cdot u\;\; d\Omega > 0, \forall u \qquad (3.1.5)$$
$$\int_\Omega \mathrm{L}\; u \;\cdot u\;\; d\Omega = 0 \Leftrightarrow u \equiv 0$$

To determine whether LΩ is positive-defined, we can integrate the dot product into parts until it contains derivatives of the same order. This operation is fundamental in the transformation of LΩ into L^*.

It should always be borne in mind that the defined positivity property is extremely important for the establishment of solution schemes and also in the construction of variational procedures.

Examples

1 Properties analogous to the self-adjoint and positive-defined in LΩ operators can also be defined for matrices (as presented in chapter 1); these properties are respectively symmetry and positive-defined. (Remember that a matrix is said to be symmetric if $\mathbf{A} = \mathbf{A}^T$ or even $y, \mathbf{A}x = x, \mathbf{A}y$ and is said to be positive-de$x^T\mathbf{A}x > 0, \forall x$ e $x^T\mathbf{A}x = 0 \Leftrightarrow x = 0$).

2 Show that the operator $\mathrm{L} = -\frac{d^2}{dx^2}$ is self adjoint and positive defined in the interval [0,1].

Solution:

Let you and v be any two functions, then

$$(\mathcal{L}(u).v) = \int_0^1 \left(-\frac{d^2u}{dx^2} \right) v dx =$$
$$= -\left| \frac{du}{dx} v \right|_0^1 + \int_0^1 \frac{du}{dx}\frac{dv}{dx} =$$
$$= -\left| \frac{du}{dx} v \right|_0^1 + \left| u \frac{dv}{dx} \right|_0^1 + \int_0^1 \left(-\frac{d^2v}{dx^2} \right) u dx$$

that comparing with (12.3.4) we see that Ω is self-adjunct. Note that:

$$\mathbf{G}\,u = u;$$
$$\mathbf{S}\,u = -\frac{du}{dx}$$
$$\mathbf{G}\,v = v;$$
$$\mathbf{S}\,v = -\frac{dv}{dx}$$
$$\mathbf{S} = \mathbf{S}^*$$

The essential boundary condition u is prescribed; and the natural boundary condition is -du/dx prescribed. If $u=v$ and boundary conditions are homogeneous, it will be seen that $\mathcal{L}_\Omega$ it is positive-defined.

Investigate the properties of the operator $\mathcal{L}_\Omega\, u = \frac{d^4u}{dx^4}$ in [0,1].

Solution

Let v be any auxiliary function, then the dot product is:

$$\int_0^1 \mathcal{L}(u)vdx = \int_0^1 \frac{d^4u}{dx^4} vdx$$

Integrating the above expression four times, we have:

$$\int_0^1 \mathcal{L}(u)vdx = \left| v\frac{d^3u}{dx^3} \right|_0^1 - \left| \frac{dv}{dx}\frac{d^2u}{dx^2} \right|_0^1 + \left| \frac{d^2v}{dx^2}\frac{du}{dx} \right|_0^1 - \left| \frac{d^3v}{dx^3} u \right|_0^1 + \int_0^1 \frac{d^4v}{dx^4} u dx$$

Putting the above expression in the form of (12.3.4) it is seen that:

$$\int_0^1 \mathcal{L}(u)v\,dx = \int_0^1 \mathcal{L}^*(v)u\,dx + \left[G_1(v)S_1(u) - G_2(v)S_2(u)\right]_0^1 + \left[S_2(v)G_2(u) - S_1(v)G_1(u)\right]_0^1$$

Therefore, the essential boundary conditions are:

$$G_1(u) = u; G_2(u) = \frac{du}{dx}$$

Prescribed and natural boundary conditions are:

$$S_1(u) = \frac{d^3u}{dx^3}; S_2(u) = \frac{d^2u}{dx^2},$$ Prescribed.

The operator $\mathcal{L}^*(\bullet) = \frac{d^4(\bullet)}{dx^4} = \mathcal{L}(\bullet)$ => be self-adjoint.

Now by placing *u*=*v* and making the boundary conditions homogeneous, we get:

$$\int_0^1 \mathcal{L}(u)u\,dx = \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx \geq 0$$

which shows that the operator in question is positive-defined. That is, so that for all $\mathcal{L}_\Omega(u) \geq$ and any function u , it must be constrained to at least u0=u1=0 or

$$\left(u(0) = 0 \wedge \left(u(1) = \frac{du}{dx} = 0\right)\right).$$

3.3 THE WEIGHTED RESIDUES METHOD - GENERAL METHOD

Weighted residuals methods are numerical procedures for the solution of a set of differentiable equations of the form:

$$\mathcal{L}(u_0) = p \text{ em } \Omega \tag{3.1.6}$$

with the following boundary conditions:

essential: $\mathcal{L}\ u_0 \ = g|_{\Gamma_1}$

natural: $\mathcal{L}\ u_0 \ = q|_{\Gamma_2}$

with $\Gamma_1 + \Gamma_2 = \Gamma$ the outer surface of the domain being Ω , and u0 being the exact solution. The function u0 is first approximated by functions $\phi_k\ x$, such that:

$$u = \sum_{k=1}^{n} \alpha_k \phi_k \tag{3.1.7}$$

where αk are indeterminate parameters and $\phi_k\ x$re linearly independent functions, taken from a complete sequence of functions $\phi_1, \phi_2 \dots \phi_n$. These functions are usually chosen in such a way that they satisfy certain conditions, called admissibility conditions, relating the boundary conditions and the degree of continuity. Functions are considered to belong to a linear space, that is, they can be combined linearly and have a dot product, norm, and metric as defined in chapter 2 of volume 1 of this work.

Remembering that a sequence of linearly independent functions is said to be complete (see section 2.6 et seq. in volume 1 of this work) if a number N and a set of constants α can be found such that, given an admissibility and an arbitrary function u_0 , we have:

$$\left\| u_0 - \sum_{i=1}^{N} \alpha_i \phi_i \right\| \leq \beta \tag{3.1.8}$$

where the amount β is as small an amount as you want.

The functions ϕ for the problem (2.3.6) must satisfy the conditions (2.3.7) and must necessarily have a sufficient degree of continuity to make the left-hand side of (2.3.7) nonzero.

Substituting (12.3.8) into (12.3.6) we find an error function €, called a residual, i.e.:

$$\varepsilon = \mathcal{L}_\Omega\ u\ - p \neq 0 \tag{3.1.9}$$

Note that € is equal to zero for the exact solution, but not for an approximate solution. The residual is forced to zero in the mean sense, i.e., on average, by means of a procedure of zeroing the weighted integral of the residual:

$$\int \varepsilon \psi_i dx = 0, \forall i = 1,2,\ldots,n \tag{3.1.10}$$

Where ψ_i a set of weighting functions, which are also part of a complete and linearly independent set.

By doing so, the solution converges to the exact solution, with an increase in the number of terms adopted.

NOTE: It is good to remember that an infinite set of orthogonal functions is not necessarily complete, that is, one can have an infinite number of functions, but the solution does not converge to the exact solution. For a sequence of functions to be complete, it is necessary and sufficient that each and every subsequence of the sequence be a Cauchy sequence. That converges to the same function in the space that contains them. Thus, a metric space is complete when all Cauchy sequences converge to a boundary that belongs to space.

Be the following set of functions $\phi_k = \sin\left(\frac{k\pi x}{l}\right), \forall k = 1,2,\ldots,n$ and be the solution.

$$u = \sum_{k=1}^{\infty} \alpha_k \phi_k$$

This does not allow the *solution U* = constant to be reproduced . If the possibility of *u* being constant exists, then there is a need to have a term in the solution that allows the desired solution to be reproduced. Like this

$$u = \alpha_0 \cdot 1 + \sum_{k=1}^{\infty} \alpha_k \phi_k$$

which is now a complete set. This example illustrates the difficulty in establishing the completeness of a given set of functions.

The following are some methods based on the idea of orthogonalization. In them, the weighting functions are chosen in different ways. In principle, only self-adjoint and positive-defined operators will be considered for the sake of simplicity of presentation of the methods, but the same applies to operators of more general types, which will be seen later.

3.4 METHOD OF MOMENTS.

This method was developed by H. Yamada in 1947 and H. Fujita in 1951 for application to laminar boundary layer and nonlinear transient diffusion problems, respectively. As we have mentioned earlier, the weighting functions can be differen ψ_i rom the approximation functions ϕ_i can be used.

A simple choice is the LI and complete set:

$$1, x, x^2, x^3, \dots \qquad (3.1.11)$$

for one-dimensional problems. In this way, successive moments of increasing order of the residual are forced to zero like this:

$$\begin{gathered} \int \varepsilon \psi_i dx = \int \mathrm{L}\; u \;\; -p \;\; \psi_j dx = 0 \\ \therefore u = \sum \alpha_i \phi_i \\ \psi_j = x^j, \forall j = 0,1,2,\dots,n \end{gathered} \qquad (3.1.11)$$

The above technique is called the **<u>Method of Moments</u>**, because of the type of choice made for the weighting functions. If any other set is chosen for the weighting functions, it will no longer be the Method of Moments, but only a weighted residue method.

Example

Let $\varepsilon = ◆\; u \;\; -p = \dfrac{d^2u}{dx^2} + u + x = 0$ Ω, where Ω = 0.1 , with the following conditions of contour $u\;\; 0 \;\; = u\;\; 1 \;\; = 0$.

Be the following approach for the solution:

$$u = x\;\; 1 - x \quad \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots$$

that satisfies the given boundary conditions. For the purpose of calculation, only the first two terms in α will be considered, i.e.:

$u = x \;\; 1 - x \quad \alpha_1 + \alpha_2 x$

So the residual function will be:

$\varepsilon = \blacklozenge \; u \quad - p = x + \; -2 + 39x - x^2 \;\; \alpha_1 + \; 2 - 6x \; + x^2 - x^3 \;\; \alpha_2$

Orthogonalizing the residual with respect to the weighting functions 1 and x, we have, respectively:

$$\int_0^1 \varepsilon \cdot 1 dx = 0$$
$$\int_0^1 \varepsilon \cdot x dx = 0$$

By integrating, a system is obtained, which placed in matrix form is:

$$\begin{bmatrix} \frac{11}{6} & \frac{11}{12} \\ \frac{11}{12} & \frac{19}{20} \end{bmatrix} * \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{Bmatrix}$$

whose solution is:

$$\alpha_1 = \frac{122}{649} ; \alpha_2 = \frac{110}{649}$$

Thus the approximate solution function is $u \;\; = x \;\; 1 - x \left(\frac{1}{649} \right) \; 122 + 110x$

The exact solution of the problem posed is $u_{exata} = \left(\frac{\sin \; x}{\sin \; 1} \right) - x$

Using Mathcad you have:

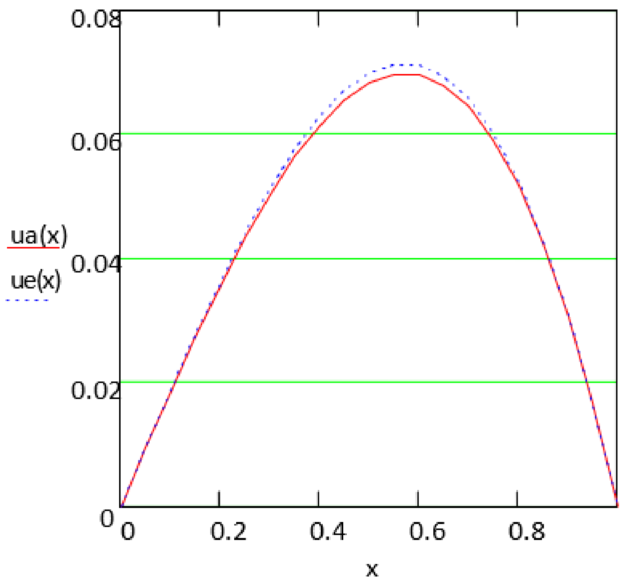
$x := 0, 0.05 .. 1$

$$ua(x) := x \cdot (1 - x) \cdot \left(\frac{1}{649}\right) \cdot (122 + 110 \cdot x)$$

$$ue(x) := \left(\frac{\sin(x)}{\sin(1)}\right) - x$$

x =	ua(x) =	ue(x) =
0	0	0
0.05	$9.332 \cdot 10^{-3}$	$9.395 \cdot 10^{-3}$
0.1	0.018	0.019
0.15	0.027	0.028
0.2	0.036	0.036
0.25	0.043	0.044
0.3	0.05	0.051
0.35	0.056	0.057
0.4	0.061	0.063
0.45	0.065	0.067
0.5	0.068	0.07
0.55	0.07	0.071
0.6	0.07	0.071
0.65	0.068	0.069
0.7	0.064	0.066
0.75	0.059	0.06

Figure 2 u''+u+x=0 solution



3.5 PLACEMENT METHOD

This method appeared in the solution of differential equations performed by J. C. Slater in 1934 and by J. Barta in 1937, respectively applied to the solution of electronic energy problems in torque mats in prismatic parts of square section. It was later generalized by R. A. Frazer (1937) and C. Lanczos (1938).

This method consists of nullifying the residual function □ at a series of chosen points within the integration domain of the problem. It should be noted that these points are usually, but not necessarily, distributed across the domain.

Let the following approximation function then be:

$$u = \sum \alpha_k \phi_k \tag{3.1.12}$$

where ϕ_k satisfies the boundary conditions.

Now be determine the values of αk by forcing the condition:

$$\varepsilon = \mathcal{L}\ u\ - p = 0 \tag{3.1.13}$$

Let Δ be the Dirac delta function, a function that is equal to zero if $x \neq x_k$, e $\int_{x_k - c}^{x_k + c} \Delta(x_k)dx = 1$,

when c ⟶ 0. Thus, one can write the placement method as a weighted residue technique:

$$\int_{\Omega} \varepsilon \Delta_k dx = 0, \forall k = 1,2,\ldots,n \tag{3.1.14}$$

Examples

1. Be solve the differential equation $\frac{d^2u}{dx^2} + 4u - 4x = 0$ *in Ω [0,1]*. Consider x *= 0.25* and x *= 0.5.*

Solution

A – Let $u_1 = \alpha\ x - x^2$ be admissible function taken from the permissible x^i, x^{i+1} set, which are LI. Using *u1* and making $\varepsilon\big|_{x=0,5} = 0$ it has.

$$\varepsilon\big|_{x=0,5} = \left[-2 + 4\ x - x^2\ \right]\alpha - 4x\ \big|_{x=0,5} = 0$$

solving is α = - 1, so $= -1 \ x - x^2$

b) Be now. $u_2 = \alpha \ x - x^2 \ + \alpha_2 \ x^2 - x^3$ By doing, we have $\varepsilon\big|_{x=0{,}25} = 0 \ \wedge \ \varepsilon\big|_{x=0{,}5} = 0$

$$
\begin{aligned}
-1{,}7500\alpha_1 - 0{,}8125\alpha_2 &= 2 \\
-1{,}8125\alpha_1 + 0{,}5469\alpha_2 &= 1
\end{aligned}
$$

$$\alpha_1 = -0{,}7846 \ \wedge \ \alpha_2 = -0{,}7717$$

Soon

$$u^{MC}{}_2 = -0{,}7846 \ x - x^2 \ - 0{,}7717 \ x^2 - x^3$$

which shows that the approximate solution in this method depends on the placement points. Using Mathcad you have:

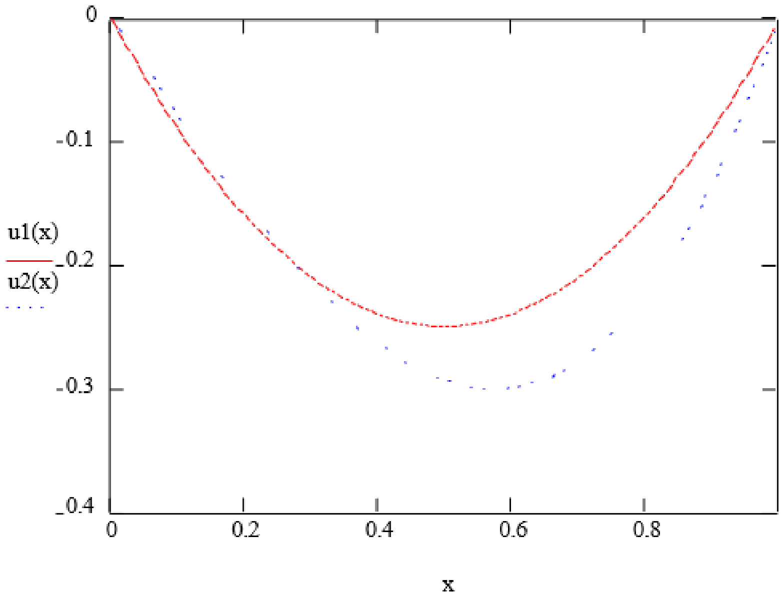
Placement Method for the equation $u'' + 4u - 4x = 0$

x:=0,0.01..1

$u1(x) := -\left(x - x^2\right)$

$u2(x) := -0.7846\left(x - x^2\right) - 0.7717\left(x^2 - x^3\right)$

Figure 3 Solution of u"+4u-4x=0 by Placement Method



2. Let us solve the following Poisson differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = p$$

with the following boundary conditions: $u(x,y) = 0, \forall (x = \pm a \wedge y = \pm b)$.

Solution

Let the following approximation function be:

$$u = (x^2 - a^2)(y^2 - b^2)\left[\alpha_1 + \alpha_2 (x^2 + y^2) + \cdots\right]$$

which is general. To simplify the example, let us consider only the first term of the sum, and a square region a $= b$. In this way, we have:

$$u_1 = \alpha_1 (x^2 - a^2)(y^2 - a^2)$$

soon

$$\varepsilon = \nabla^2 u - p = 2\left[(y^2 - a^2) + (x^2 - a^2)\right]\alpha_1 - p$$

Doing lies: $\varepsilon|_{x=y=0} = \varepsilon|_{x=y=a/2} = 0$

$$\alpha_1 = -\left(\frac{19}{60}\right)\left(\frac{p}{a^2}\right); \alpha_2 = -\left(\frac{1}{15}\right)\left(\frac{p}{a^2}\right)$$

Soon

$$u_2 = x^2 - a^2 \quad y^2 - a^2 \left(\frac{1}{15}\right)\left(\frac{p}{a^2}\right)\left(-\left(\frac{19}{4}\right) - \left(\frac{1}{a^2}\right) x^2 + y^2\right)$$

Solution in Mathcad:

$a := 1 \; b := 1 \; p := 1$

$i := 1..20j := 1..2$

$x_i := -a + 0.05 \cdot iy_j := -b + 0.05 \cdot j$

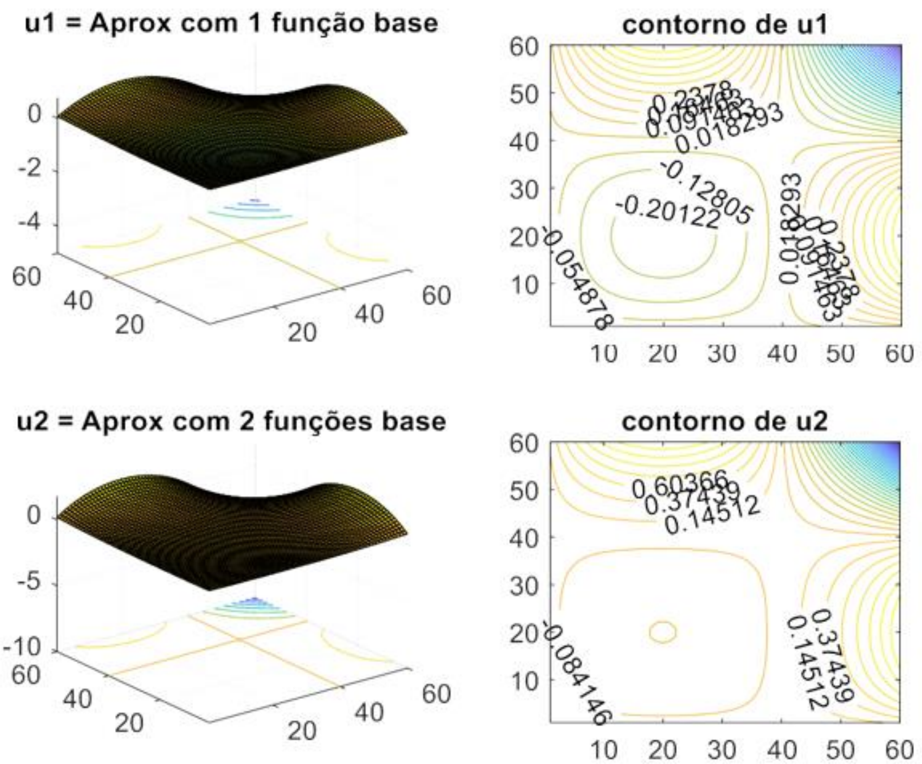
$$u1(x, y) := -\frac{p}{4 \cdot a^2} \cdot \left(x^2 - a^2\right) \cdot \left(y^2 - b^2\right)$$

$$u2(x, y) := -\left(x^2 - a^2\right) \cdot \left(y^2 - b^2\right) \cdot \left[19 \cdot \frac{p}{60 \cdot a^2} + \frac{1}{15} \cdot \frac{p}{a^4} \cdot \left(x^2 + y^2\right)\right]$$

$A_{i,j} := u1\left(x_i, y_j\right) \; B_{i,j} := u2\left(x_i, y_j\right)$

```
Solução em Matlab
function exemp2
% exemplo do capitulo 12 secao 12.3.5 método da colocação
    clc;
    global a b p
    a = 1;
    b = 1;
    p = 1;
    x = -a+0.05:0.05:2;
    y = -b+0.05:0.05:2;
    [~,~] = meshgrid(x,y);
    [nx,~] = size(x');
    [~,my] = size(y);
    Z1 = zeros(nx,my);
    Z2 = zeros(nx,my);
    for i=1:nx
        for j=1:my
            Z1(i,j) = u1(x(i),y(j));
            Z2(i,j) = u2(x(i),y(j));
        end
    end
    figure(1)
    surf(Z1);
    figure(2)
    surf(Z2);
    figure(3)
    contour(Z1,30,'ShowText','on')
    figure(4)
    contour(Z2,30,'ShowText','on')

    function [A] = u1(X,Y)
        A = -p*(X^2-a^2)*(Y^2-b^2)/(4*a*a);
    end
    function [B] = u2(X,Y)
        B = -(X^2-a^2)*(Y^2-b^2)*(19*p/60*a^2+(1*p/15*a^4)*(X^2+Y^2));
    end
end
```

Figure 4 One- and Two-Term Pair Solution



3.6 SUB-REGIONS METHOD

This method first appeared in 1923 by the German engineers C. B. Biezeno and R. Koch to solve problems arising from the stability of beams and plates.

This method is similar to the placement method described above, but here instead of zeroing the error function at certain points, we try to zero the residual function over small regions of the domain, i.e., dividing the domain into small regions, canceling out the error integral over each region:

$$\int_{\Omega} \varepsilon d\Omega = 0 \qquad (3.1.15)$$

for different Ωi and with $\bigcup \Omega_i = \Omega$ and more $\Omega_i \cap \Omega_j = \varnothing, \forall i \neq j$

Example

Be resolve $\frac{d^2u}{dx^2} + 4u - 4x = 0$ to [0,1] by splitting the domain into two regions.

$$\left\{0 \leq x \leq \frac{1}{2}\right\} \wedge \left\{\frac{1}{2} \leq x \leq 1\right\}.$$

Solution

Be. $u_2 = \alpha_1 \; x - x^2 \; + \alpha_2 \; x^2 - x^3$ Zeroing out the waste in each sub-region, we have:

$$\int_0^{1/2} \varepsilon dx = \int_0^{1/2} \; 4x^2 - 4x - 2 \; \alpha_1 + \; 4x^3 - 4x^2 - 6x + 2 \; \alpha_2 - 4x \; dx = 0$$
$$\int_{1/2}^{1} \varepsilon dx = \int_{1/2}^{1} \; 4x^2 - 4x - 2 \; \alpha_1 + \; 4x^3 - 4x^2 - 6x + 2 \; \alpha_2 - 4x \; dx = 0$$

that solving, it is found:

$$\alpha_1 = -0{,}44230$$
$$\alpha_2 = -0{,}61538$$

Therefore, the approximation function, by the sub-region method, is:

$u^{SR}{}_2 = -0,44230\ x - x^2 - 0,61538\ x^2 - x^3$

3.7 LEAST SQUARES METHOD

This method appeared in 1795 with the mathematician Johann Carl Friedrich Gauss when he studied the estimation of curve fitting by the least squares method.

In this method, the weighting functions ψ_i are chosen as follows:

$$\psi_i = \mathcal{L}\ \phi_i \tag{3.1.16}$$

Where ϕ_i are LI functions and part of a complete set. Let then , where ◆ $u\ = p$ *is* an approximation function of u , with

$$u_1 = \sum \alpha_i \phi_i \tag{3.1.17}$$

So the residual will be:

$$\varepsilon = \mathcal{L}\ u_1\ - p \tag{3.1.18}$$

Since we must have the dot product between the residual € and the weighting functions equal to zero in the domain, we have:

$$\varepsilon, \psi_i\ = \int_\Omega \varepsilon \psi_i d\Omega \tag{3.1.19}$$

As the functions, $\psi_i\ = \mathcal{L}\ \phi_i$ substituting in the above expression, and resolving comes:

$$\varepsilon, \psi_i\ = \mathcal{L}\ u_1\ - p, \psi_i\ = \mathcal{L} \sum \alpha_i \phi_i\ - p, \mathcal{L}\ \phi_i |$$
$$= \alpha_i\ \mathcal{L}\ \phi_i\ , \mathcal{L}\ \phi_i\ - p, \mathcal{L}\ \phi_i\ = 0$$

As , we l $\mathcal{L}\ \phi_i\ , \mathcal{L}\ \phi_i\ = \int \mathcal{L}\ \phi_i\ \mathcal{L}\ \phi_i\ = \|\mathcal{L}\ \phi_i\ \|^2$

$$\alpha_1 \|\cdots\ \phi_i\ \|^2 - p, \cdots\ \phi_i\ = 0, \forall i = 1,2,\ldots,n \tag{3.1.20}$$

ψ

Example

Let be solving the same problem as in the previous section, that is, solving $\frac{d^2u}{dx^2} + 4u - 4x = 0$ *at [0,1].*

Solution:

Be , s $u_2 = \alpha_1 \ x - x^2 \ + \alpha_2 \ x^2 - x^3 \ \therefore \phi_1 = \alpha_1 \ x - x^2 \ , \phi_2 = \alpha_2 \ x^2 - x^3$

$$\alpha_1 \| L \ \phi_1 \ \|^2 - \ p, L \ \phi_1 \ = 0$$
$$\alpha_2 \| L \ \phi_2 \ \|^2 - \ p, L \ \phi_2 \ = 0$$

whose solution is

$$\begin{cases} \alpha_1 = -\frac{110}{101}; \\ \alpha_2 = -\frac{399}{449} \end{cases}$$

and

$$u^{MQ}{}_2 = -\left(\frac{110}{101}\right) \ x - x^2 \ - \left(\frac{399}{449}\right) \ x^2 - x^3$$

Comparing u_2^{MC} (solution by the placement method), with u_2^{SR} (from the example in the previous section, by the method of sub-regions) and u_2^{MQ} (solution by the method of least squares), we have:

$$u^{MC}{}_2 = -0,7846 \ x \ - x^2 \ - 0,7717 \ x^2 - x^3$$

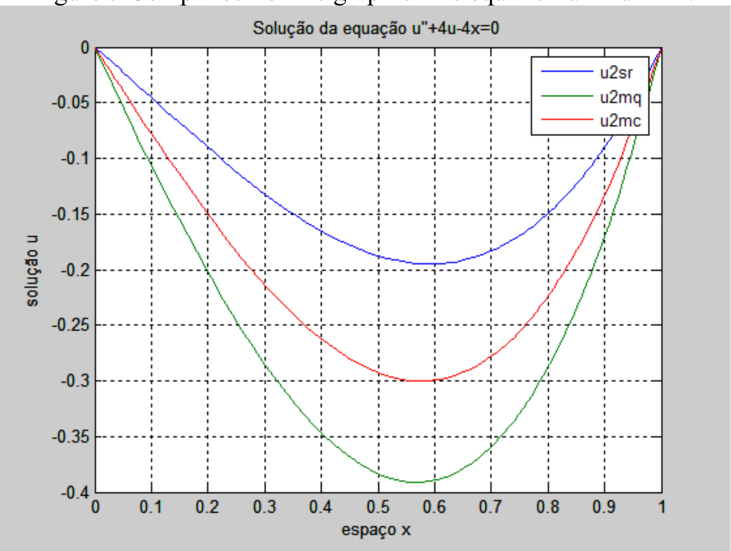
$$u^{SR}{}_2 = -0,44230 \ x - x^2 \ - 0,61538 \ x^2 - x^3$$

$$u^{MQ}{}_2 = -\left(\frac{110}{101}\right) \ x - x^2 \ - \left(\frac{399}{449}\right) \ x^2 - x^3$$

Making use of a small procedure in Matlab:

```
clear;clc;
% função auxiliar do ODE solver ode45
vdu  = @(x,y)[y(2); -4*y(1)-4*x];    % u1' = u2
% u2' = -4u1 - 4x
% solução pelo método da colocação
u2mc = @(x)-0.7848*(x-x^2)-0.7717*(x^2-x^3);

% solução pelo método das sub-regiões
u2sr = @(x)-0.4430*(x-x^2)-0.61538*(x^2-x^3);

% solução pelo método dos mínimos quadrados
u2mq = @(x)-(110/101)*(x-x^2)-(399/449)*(x^2-x^3);

% plotar as funções
fplot(@(x)[-0.4430*(x-x^2)-0.61538*(x^2-x^3),...
-(110/101)*(x-x^2)-(399/449)*(x^2-x^3),...
    -0.7846*(x-x^2)-0.7717*(x^2-x^3)], [0,1]); grid on;holdon;
title('Solução da equação u"+4u-4x=0');
xlabel('espaço x');
ylabel('solução u');
legend('u2sr','u2mq','u2mc');
```

Figure 5 Comparison of the graph of the equation u"+4u-4x=0

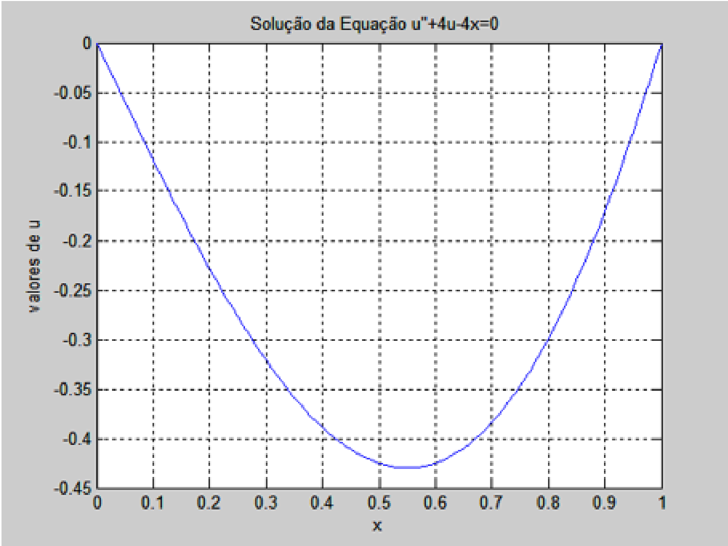


The solution of the differential equation found using Matlab using the bvp4c function is shown in the procedure below (commented to facilitate the reader's understanding) and whose result is presented in figure 2:

```
function mat4bvp_mod
%MAT4BVP_MOD  Acha a solução da equação:
%
%      u'' + 4u - 4x = 0
%
%   sobre o intervalo [0, 1] com as seguintes condições de contorno:
%      u(0) = 0, u(1) = 0
%
%   Primeiramente se utiliza a função bvpinit para se determinar uma
%      condição inicial para se poder resolver o problema de valores
```

```
%      decontorno pelo solver bvp4c.

%   solinit = bvpinit(x,yinit) onde x é um vetor que especifica

%      uma malha inicial e yinit é em geral uma função com os

%      valores iniciais de y nas abscissas x.


clc;

solinit = bvpinit(linspace(0,1,21),@mat4init);

% solinit.x     % mesh dada por linspace acima
%   sol.x          Malha selecionada por bvp4c
    sol.y
%   sol.yp         Aproximação para y(x) nos pontos da malha de sol.x
    sol.parameters Valores retornados por bvp4c para parametros
%                  desconhecidos, se existirem

%   sol.solver     'bvp4c'

%   Se queremos resolver o problema de valores de contorno (BVP)
       sobre [a,b], então especificamos x(1) como 'a' e x(end)
%      como 'b', e colocamos isso num vetor coluna que pode

%      ser referenciado por uma função 'bcfun'

sol = bvp4c(@mat4ode,@mat4bc,solinit);


% Agora se pode listar ou plotar os valores da solução nos pontos

%   da malha entre os pontos do intervalo definido.

% Em geral, a solução aproximada S(x) é contínua e tem

%   derivadas continuas. Pode-se utilizar a função

% DEVAL para avaliar bastante pontos para se ter um gráfico suave.


xint = linspace(0,1,101);
Sxint = deval(sol,xint);

%Sxint(1,:) = valor da solução em x = Sxint(2,:)
figure;

dydx = [    y(2)                    % u" =   y'    <=> y(1)  = y(2)

% --------------------------------------------------------------------------------------------

function res = mat4bc(ya,yb)

res = [   ya(1)                     % ya(1) = y'(a) --> ya(1) = 0

yb(1) ];                   % yb(1) = y'(b) --> yb(1) = 0

%
```

Figure 6 Graph of u"+4u-4x=0 solution by matlab solver bpv4c



3.8 GALERKIN'S METHOD

Due to the importance of Galerkin's method, we will not describe it here, but will open a section just to discuss it. This will be done in the next section.

4 GALERKIN'S METHOD

4.1 GENERAL

The Galerkin method is a particular case of the weighted residuals method, in which the weighting functions are ψ_i ɜ same as the approximation functions ϕ_i .

This method approximates the solution of a given set of differential equations and their boundary conditions by substituting in them one or more verification functions, which, in principle, satisfy the boundary conditions. Since the verification functions are generally different from the exact solution, the set of equations produces some residues. These residuals are then weighted by the approximate solution modes and made equal to zero, over the domain.

Let the system of equations be:

$$\varepsilon\big|_{\Omega} = \cdots\ u_1 \ - p = 0 \tag{4.1.1}$$

with the essential and natural boundary conditions on ◆ $u\ \big|_{\Gamma} = g;$ ◆ $u\ \big|_{\Gamma} = q$ about Γ,
where $\cdots$, ◆, ◆ they can be differentiable, integral, integrative-differentiable operators, etc.

Let *u1 be* an approximation function that satisfies the boundary conditions and formed by the complete combination ϕ_i of functions:

$$u_1 = \sum_{i=1}^{n} \alpha_i \phi_i \tag{4.1.2}$$

with residue

$$\varepsilon|_{\Omega} = \cdots u_1 \; - p = \cdots \left(\sum_{i=1}^{n} \alpha_i \phi_i \right) - p \tag{4.1.2}$$

that must be orthogonalized with respect to the same approximation function ϕ_k that is:

$$\varepsilon, \phi_k \; = \int_{\Omega} \varepsilon \phi_k d\Omega = 0, \forall k = 1,2,\ldots,n \tag{4.1.3}$$

If the operator is linear, the above equation yields a system of linear equations from which the coefficients αk can be obtained:

$$\varepsilon, \phi_k \; = \int_{\Omega} \cdots \; \phi_i \quad \phi_k d\Omega = \int_{\Omega} p \phi_k d\Omega, \forall i,k = 1,2,\ldots,n \tag{4.1.4}$$

or in matrix form

$$\mathbf{A}\alpha = p \tag{4.1.5}$$

Of course, the approximation function *u1* belongs to the space *H* generated by the functions $\phi_i \in H$ Thus, the Lax–Milgram theorem establishes the convergence criterion of the process.

Before enunciating it, it is necessary to define what is a bounded differential operator and a coercive differential operator.

Definition: A linear differential operator is said to be bounded operator if there is $\delta > 0$ such that

$$\begin{gathered} \left| \mathrm{L} \; v \;, w \; \le \delta \|v\|_{\mathrm{H}} \times \|w\|_{\mathrm{H}}, \quad \forall v,w \in \mathrm{H} \right| \\ \therefore \|v\|_{\mathrm{H}} = \left[\|v\|^2 + \; \cdots \; v \;, v \; \right]^{1/2} \end{gathered} \tag{4.1.6}$$

Definition: A linear differential operator is said to be a coercive operator if there exists $\kappa > 0$ such that

$$\begin{gathered} \left| \cdots \; v \;, v \; \le \kappa \|v\|_{\neg}^{2}, \quad \forall v \in_{\neg} \right| \\ \therefore \|v\|_{\neg} = \left[\|v\|^2 + \; \cdots \; v \;, v \; \right]^{1/2} \end{gathered} \tag{4.1.7}$$

Theorem: (Lax-Milgram). Be a bounded and coercive linear differential operator; let V be a subspace of ¬. So there is a single *u1* belonging to V such that

$$\cdots u_1 - p, v = 0, \quad v \in V \tag{4.1.8}$$

and more,

$$\cdots u_1 - p, v = 0, \quad v \in V \tag{4.1.9}$$

Where $\phi_0 \in ¬$ is arbitrary and *you* is a weak solution $\cdots u = p$ of with x € Ω in ¬ Ω.

Definition: The inequality (12.4.10) that appears in the Lax–Milgram theorem is called **Cea's lemma**:

$$\|u_1 - u\|_{¬} \leq \frac{\delta}{\kappa} \inf \|v - u\|_{¬} : v \in \phi_0 + V$$

Examples

1. Solve the equation $\frac{d^2u}{dx^2} + u + x = 0$ at [0,1] and with the conditions $u \; 0 = u \; 1 = 0$.

Solution

Since the approximation function has to satisfy the boundary conditions, we have:

$$u^* = x \; 1 - x \quad \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots$$

Be it as a first approximation , $u_1 = x \; 1 - x \quad \alpha_1 + \alpha_2 x$,where

$$\int_0^1 \varepsilon . \phi_k dx = 0 \therefore \varepsilon = \mathcal{L} \; u - p, \quad com \begin{cases} \phi_1 = x \; 1 - x \\ \phi_2 = x^2 \; 1 - x \end{cases}$$

Soon

$$\int_0^1 \varepsilon.\phi_1 dx = \int_0^1 \varepsilon.x.\ 1-x\ \ dx = 0$$
$$\int_0^1 \varepsilon.\phi_2 dx = \int_0^1 \varepsilon.x^2.\ 1-x\ \ dx = 0$$

Integrating, we find:

$$\alpha_1 = \frac{71}{369};$$
$$\alpha_2 = \frac{7}{41};$$

that generates

$$u_1 = x\ \ 1-x\ \left(\frac{71}{369} + \frac{7}{41}x\right)$$

Comparison of the Solution by Galerkin's Method and of the Moments and Exact

x := 0□0 + 0.05□□1

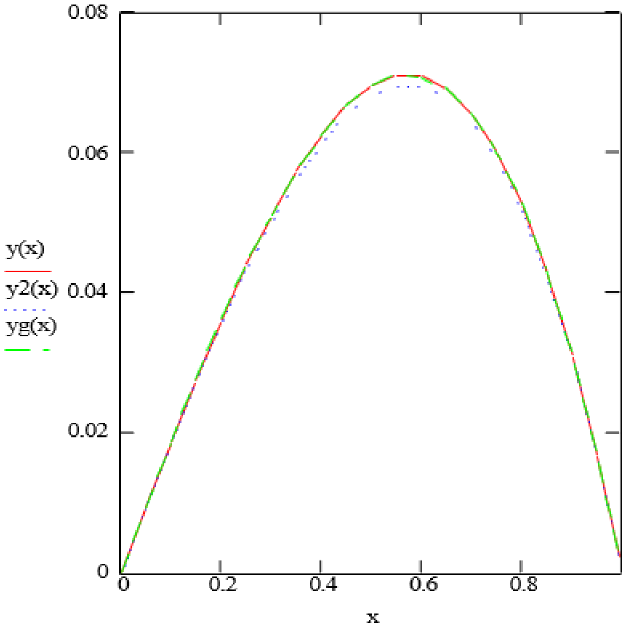
Solution by the Method of Moments *y2*(x):

Solution by Galerkin's method *yg*(x):

Exact Solution *y(x):*

$$y2(x) := x\cdot(1-x)\cdot\left(\frac{122}{649} + \frac{110}{649}x\right) \quad yg(x) := x\cdot(1-x)\cdot\left(\frac{71}{369} + \frac{7}{41}\cdot x\right) \quad y(x) := \frac{\sin(x)}{\sin(1)} - x$$

Figure 7



x =	y 2(x) =	y g(x) =	y(x) =
0	0	0	0
0.05	$9.332\cdot 10^{-3}$	$9.545\cdot 10^{-3}$	$9.395\cdot 10^{-3}$
0.1	0.018	0.019	0.019
0.15	0.027	0.028	0.028
0.2	0.036	0.036	0.036
0.25	0.043	0.044	0.044
0.3	0.05	0.051	0.051
0.35	0.056	0.057	0.057
0.4	0.061	0.063	0.063
0.45	0.065	0.067	0.067
0.5	0.068	0.069	0.07
0.55	0.07	0.071	0.071
0.6	0.07	0.071	0.071
0.65	0.068	0.069	0.069
0.7	0.064	0.066	0.066
0.75	0.059	0.06	0.06

Find a solution to the Poisson equation with the fol $\nabla^2 u = p$ undary conditions:

$u(-a) = u(a) = u(-b) = u(b) = 0$, in a rectangular regio $[-b,b]\times[-a,a]$.

Solution:

To show the method by applying several sets, a polynomi ϕ_i solution and then a trigonometric solution will be presented.

Be $u = \alpha \; x^2 - a^2 \quad y^2 - b^2$

So $\delta u = \delta a \; x^2 - a^2 \quad y^2 - b^2$

And $\varepsilon = \nabla^2 u - p = 2\alpha \; y^2 - b^2 \; + 2\alpha \; x^2 - a^2 \; - p$.

Soon

$$\varepsilon, \phi_k \; = \int_{\Omega} \varepsilon.\phi_k d\Omega = 0, \forall k = 1,2,\ldots,n$$

$$\int_{\Omega} \varepsilon.\phi_k d\Omega = \int_{-a}^{a}\int_{-b}^{b} \varepsilon.\delta u dx dy = \int_{-a}^{a}\int_{-b}^{b} \left[\nabla^2 u - p\right] \delta u dx dy = 0$$

substituting *u* in the above expression, knowing that $\phi_1 = u$

$$\int_{-a}^{a}\int_{-b}^{b} \left[2\alpha \; y^2 - b^2 \; + 2\alpha \; x^2 - a^2 \; - p\right]. \; \alpha \; y^2 - b^2 \quad x^2 - a^2 \quad dx dy = 0$$

and by integrating, it is obtained $\alpha = \frac{5}{8}\left(\frac{p}{a^2 + b^2}\right)$

soon

$$u = \frac{5}{8}\left(\frac{p}{a^2 + b^2}\right) \; y^2 - b^2 \quad x^2 - a^2$$

Let then be the given polynomial solution:

se = (5/8)[p/(^2+b^2)] (^2-a^2) (y^2-b^2).

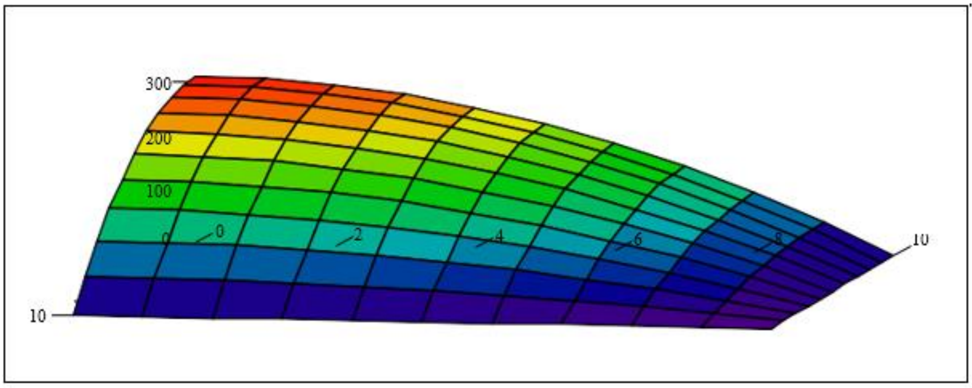
Let's consider a=1, b=1 and p varying

```
a :- 10   b :- 10

p :- 10

x :- 0..a   y :- - 0..b
```

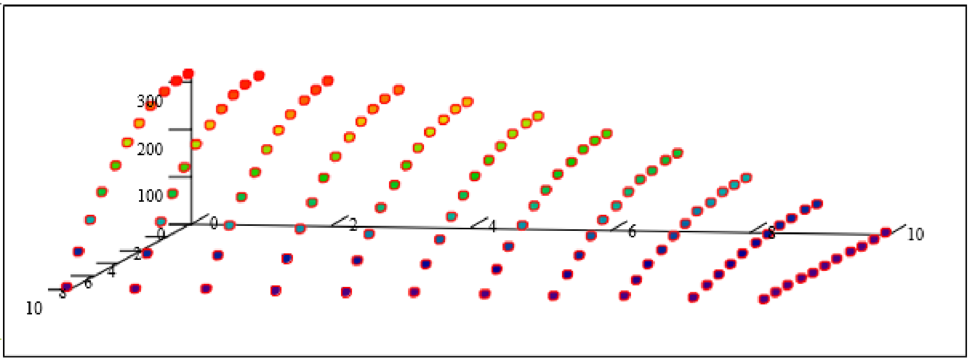
$$u_{x,y} := -\left(\frac{5}{8}\right)\cdot\left[\frac{p}{\left(a^2+b^2\right)}\right]\cdot\left(x^2-a^2\right)\cdot\left(y^2-b^2\right)$$

u =

	0	1	2	3	4	5	6	7	8
0	312.5	309.375	300	284.375	262.5	234.375	200	159.375	112.5
1	309.375	306.281	297	281.531	259.875	232.031	198	157.781	111.375
2	300	297	288	273	252	225	192	153	108
3	284.375	281.531	273	258.781	238.875	213.281	182	145.031	102.375
4	262.5	259.875	252	238.875	220.5	196.875	168	133.875	94.5
5	234.375	232.031	225	213.281	196.875	175.781	150	119.531	84.375
6	200	198	192	182	168	150	128	102	72
7	159.375	157.781	153	145.031	133.875	119.531	102	81.281	57.375



u



u

B- Let be now $u = \sum_k \sum_j \alpha_{jk} \cos\left(\frac{k\pi x}{2a}\right)\cos\left(\frac{j\pi y}{b}\right)$ a function that satisfies the boundary conditions and belongs to a set LI, i.e.:

$$\int_{-a}^{a} \cos\left(\frac{k\pi x}{2a}\right)\cos\left(\frac{j\pi y}{2b}\right) = a\delta_{jk}$$

Making , $\phi_k = \cos\left(\frac{k\pi x}{2a}\right)\cos\left(\frac{j\pi y}{2b}\right)$ then the dot product, € = ϕ = 0 i.e.

$$\int_{\Omega} \phi\nabla_2 u - p\phi \; d\Omega = 0$$

Substituting the expressions of the given functions, integrating and solving the following results:

$$\alpha_{kj} = \frac{64a^2b^2p}{\pi^4 kj \;\; k^2b^2 + j^2a^2}$$

Therefore, the solution sought is:

$$u = \sum_k \sum_j \left\{\left[\frac{64a^2b^2p}{\pi^4 k.j. \;\; k^2b^2 + j^2b^2}\right].\cos\left(\frac{k\pi x}{2a}\right)\cos\left(\frac{j\pi y}{2b}\right)\right\}$$

Let then be the given polynomial solution:

$$u_{x,y} := \sum_{k=1}^{z} \sum_{j=1}^{z} \left[\left[\frac{\left(64 \cdot a^2 \cdot b^2 \cdot p\right)}{\pi^4 \cdot k \cdot j \cdot \left(k^2 \cdot b^2 + j^2 \cdot a^2\right)}\right] \cdot \left(\cos\left(k \cdot \pi \cdot \frac{x}{2 \cdot a}\right) \cdot \cos\left(j \cdot \pi \cdot \frac{y}{2 \cdot b}\right)\right)\right]$$

Let's consider a=10, b=10, and p=10. Be the solution in Mathcad below:

$\pi := 3.141$

$a := 10 \quad b := 10 \quad p := 10$

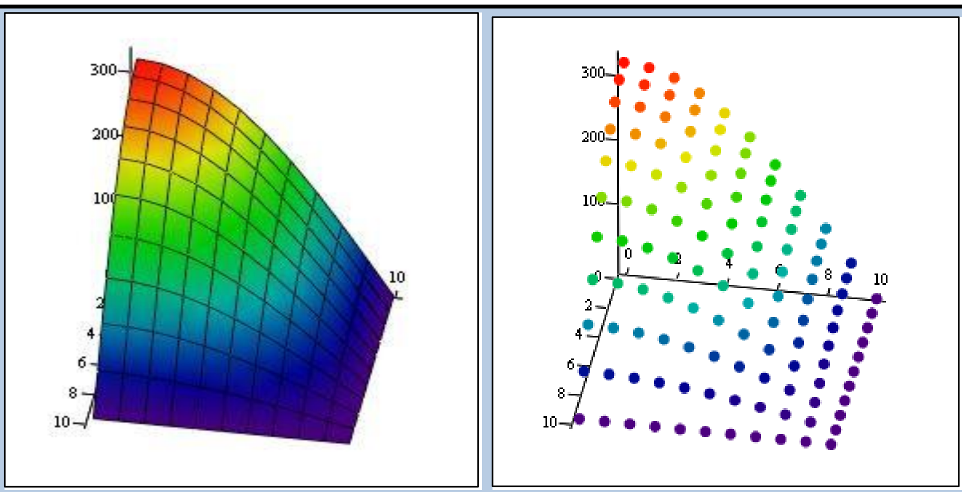
$x := 0..a \quad y := 0..b$

$z := 1$

$$u_{x,y} := \sum_{k=1}^{z} \sum_{j=1}^{z} \left[\frac{64 \cdot a^2 \cdot b^2 \cdot p}{\pi^4 \cdot k \cdot j \cdot \left(k^2 \cdot b^2 + j^2 \cdot a^2\right)^2} \right] \cdot \left(\cos\left(k \cdot \pi \cdot \frac{x}{2 \cdot a}\right) \cdot \cos\left(j \cdot \pi \cdot \frac{y}{2 \cdot b}\right) \right)$$

u =

	0	1	2	3	4
0	328.55	324.505	312.471	292.742	265.806
1	324.505	320.51	308.624	289.139	262.534
2	312.471	308.624	297.178	278.415	252.798
3	292.742	289.139	278.415	260.837	236.837
4	265.806	262.534	252.798	236.837	215.045
5	232.325	229.465	220.955	207.005	187.958
6	193.124	190.747	183.673	172.076	156.243
7	149.168	147.332	141.868	132.911	120.681
8	101.539	100.289	96.57	90.473	82.148
9	51.41	50.777	48.894	45.807	41.592
10	0.015	0.015	0.014	0.014	0.012

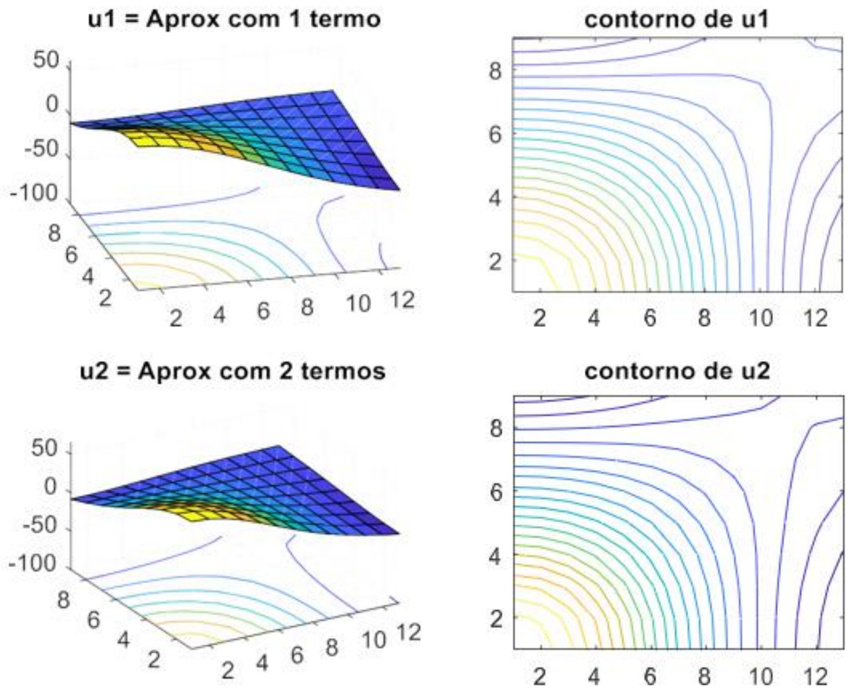


u u

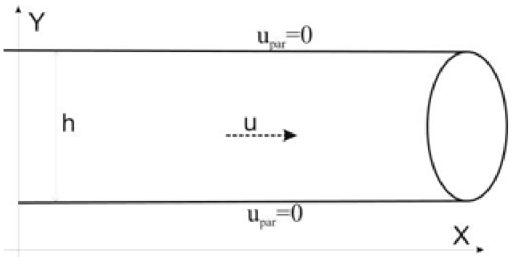
A solution in Matlab is given below for problem b): with a=4; b=6 and p=5, with a step h=0.5, generating 8 points in the x direction and 12 points in the y direction.

```
function exemp2b_mg
% exemplo 2b do capitulo 12 secao 12.4 método de Galerkin \nabla^2u=p
% em uma região retangular {|-b,b| x |-a,a|}
% face simetria, solução em {|0,b| x |0,a|}
    clc;
    global a b p
    disp('Valores Default: a=b=p=10');
    a = input('Entre com a dimensão x:  ');
    if a ==0
        a = 10;
    end
    b = input('Entre com a dimensão y:  ');
    if b == 0
        b = 10;
    end
    p = input('Entre com o termo independente:  ');
    if p == 0
        p = 10;
    end
    x = 0:0.5:a;
    y = 0:0.5:b;
    [~,~] = meshgrid(x,y);
    [nx,~] = size(x');
    [~,my] = size(y);
    Z1 = zeros(nx,my);
    Z2 = zeros(nx,my);
    for ix=1:nx
        for iy=1:my
            Z1(ix,iy) = uxy(x(ix),y(iy),3);
            Z2(ix,iy) = uxy(x(ix),y(iy),4);
        end
    end
    Z1
    Z2
    figure(1);
    subplot(2,2,1),
    surfc(Z1),  title('u1 = Aprox com 1 termo')
```

```
    subplot(2,2,3)
    surfc(Z2),title('u2 = Aprox com 2 termos')
    subplot(2,2,2)
%    contour(Z1,20,'ShowText','on'),title('contorno de u1')
    contour(Z1,20),
    title('contorno de u1')
    subplot(2,2,4)
    contour(Z2,20),
    title('contorno de u2')

    function [A] = uxy(X,Y,n)
        aux1 = 64*(a^2)*(b^2)*p;
        A = 0;
        for k=1:n
            for j=1:n
                aux2 = (pi^4)*k*j*((k^2)*(b^2)+(j^2)*(a^2));
                aux3 = cos(k*pi*X/(2*a))*cos(j*pi*Y/(2*b));
                A = A + aux1*aux3/aux2;
            end %for j
        end % for k
        % A(X,Y)
    end  % function uxy
end
```



c. Let us now consider the case of flow in a channel of unit depth when the velocity in the y-direction is zero ($v = 0$)



The continuity equation is: $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial v}$ he viscosity of the liquid $v = 0$ only For confined $u = u\ y$ flow and forced convection, the equation of momentum in the *x-direction* is:

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial \rho}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

With , $u_x = v_y = 0$

$$0 = -\frac{\partial \rho}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

Integrating twice with respect to y , we get:

$$0 = -\frac{\partial \rho}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad \text{original}$$

$$-h = -\frac{\partial \rho}{\partial x} y + \mu \frac{\partial u}{\partial y} \quad 1^{a}\text{integração em y}$$

$$-hy = -\frac{\partial \rho}{\partial x}\frac{y^2}{2} + \mu u \quad 2^{a}\text{integração em y}$$

arrumando, se tem

$$u = \frac{1}{2}\frac{h^2}{\mu}\left(\frac{\partial \rho}{\partial x}\right)\left(\frac{y^2}{h^2} - \frac{y}{h}\right)$$

which defines the flow between parallel plates, called the Poiseiville flow.

Using Galerkin's method, you get:

$$\int_0^h \left\{-\frac{\partial \rho}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}\right\} \delta u \, \delta y = 0$$

Being u of the form $u = a.sen(\pi y / h)$ *então* $\delta u = \sin(\pi y / h)$ that taking the integral above and realizing it, we find:

$$a = -\frac{4.h^2}{\pi^3.\mu}.\frac{\partial p}{\partial x}$$

Which produces the following function

$$u = -\frac{4.h^2}{\pi^3.\mu}.\frac{\partial p}{\partial x}.\sin\left(\frac{\pi.y}{h}\right)$$

Analysis

The weighted residue methods presented are difficult to apply in practice, as

Because of this, the tentative functions need to satisfy all boundary conditions - essential and natural. Functions need to fulfill these requirements because they use error minimization to satisfy the differential equilibrium equation

Exercises

1 - Consider the adjoint eigenform of Bessel's inhomogeneous equation

$$x * y'' + y' + (x^2 - 1) * (y / x) = -x^2, \text{com } 1 \leq x \leq 2$$

with $y1 = y\ 2 = 0$.ximation of y by the method of:

- Galerkin
- Placement
- Moments
- Minimum
- Square
- Compare solutions

2 - The extension of a square plate under unit force applied to the edges, reduces the solution of the biharmonic equation $\nabla^4 u = 0$. Given a square of side equal to 2, with the following boundary conditions $u_{xy} = u_{yx} = u_{xx} = 0$ e $u_{yy} = 11 - y^2$. In addition, at the contours x = 1 *and y = 1*

a - Show that the corresponding functional is $J = \int_\Omega \left[\nabla^2 u^{\;2} - 4u \right] dxdy$

b - Verify that the appropriate shape functions are defined by:

$$\Phi_n \ x,y = x^2 - y^2 \ \ y^2 - 1 \ \ 1, x^2 y^2, \ \ldots$$

c - Use the Galerkin method, with a single coefficient and show that the solution is close to that obtained by the Rayleigh-Ritz method:

$$u_1 \ x,y \approx 0,04253 \ x^2 - 1^{\;2} \ y^2 - 1^{\;2}$$

3 - For the deformation of a beam on an elastic base in dimensionless variables, it is:

$u^{iv} + u = 1$ with $u = u'' = 0$ for zero deflection and zero bending moment at the edges $x = 0$ and $x = 1$.

Choose shape functions $\sin \pi y$, $\sin 3\pi x$ to find an approximate solution.

4 - The nonlinear problem $\frac{d}{dx}[k(u).\frac{du}{dx}] = 0$ in $0 \leq x \leq 1$ represents the steady-state heat conduction on a plate with conductivity $k\ u$ and dimensionless temperature at the contours of $u\ 0 = 0$ and $u\ x = 1$. For $k\ u = 1 + u$ and polynomial shape approximation functions 1,x,x2, calculate the residual and approximate function by each weighted residuals method presented. Compare the solution you found with the exact solution $u\ x = 1 + 3x^{1/2} - 1$.

5- Be determine the deflection in a cable of $L = 10m$,om $g = 9,8\ m/s^2$, $\rho = 1\ kg/m$ and $T = 98\ N$, at positions L/2 and L/4. The equation that governs the problem is: $\frac{-d}{dx}(T.\frac{du}{dx}) = -\rho.g \quad 0 \leq x \leq L$.

a - Use Galerkin's method, using $\Phi_1\ a,x = a_1 \sin\left(\frac{\pi x}{L}\right)$.

b - Use Galerkin's method, using $\Phi_2\ a,x = a_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \sin\left(\frac{3\pi x}{L}\right)$.

c - Use Galerkin's method, using: $\Phi_3\ a,x = a_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \sin\left(\frac{3\pi x}{L}\right) + a_3 \sin\left(\frac{5\pi x}{L}\right)$.

6- In the previous problem, including the term due to the elastic foundation, the equation is $-\frac{d}{dx}(T.\frac{du}{dx}) + ku = -\rho.g$ with k=24.5 N/m.m = foundation stiffness. Use Shape Functions

polynomials of type . $\Phi_n = a_1 x + \cdots + a_n x^n, n = 1,2,\ldots,n$ Compare the values found with the exact solution:

$$u(x) = 0.4.\left(\frac{sen(L/2 - x/2) + senh(x/2)}{senh(L/2)}\right)$$

4.2 GALERKIN'S METHOD WITH PART-CONTINUOUS POLYNOMIALS

When approximation functions ϕ are continuous polynomials by parts over Ω. As in the definition of the method let be the system of equations:

$$\varepsilon|_{\Omega} = \cdots u_1 - p = 0 \qquad (4.2.1)$$

with the essential and natural boundary conditions $\mathcal{G}\ u\ |_{\Gamma} = g; \quad \mathcal{S}\ u\ |_{\Gamma} = q$ on Γ , where, as already seen above, differentiable, etc.

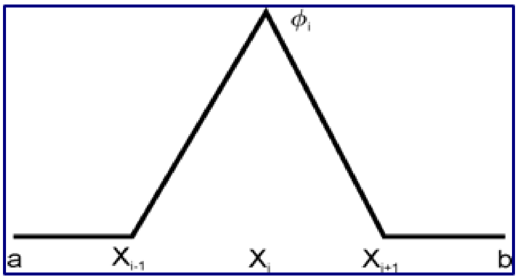
Let *u1 be* an approximation function that satisfies the boundary conditions and formed by the combination of function ϕ_i belonging to a linearly independent and complete set and more, being continuous by part defined over Ω. Without loss of generality, one can consider Ω = 0.1. Let then be a partition , $P_{\Omega_m} : 0 = x_0 < x_1 < \cdots x_m < x_{m+1} = 1$ with each subrange of length , $I_j = x_{j-1}, x_j$

so that $\Omega = \Omega_h = \phi_i^h\ x\ | x \in P_{\Omega_m}$.

$h_j = x_j - x_{j-1}$

It can be shown that Ωn is a finite vector space of dimension *m* whose basis is formed by the functions $\phi_i\ {}_{i=1}^{m}$.

Figure 8 Continuous function by part in [xi-1,xi+1]



In the normal process, all

$$u_1 = \sum_{i=1}^{n} \alpha_i \phi_i \tag{4.2.2}$$

with residue

$$\varepsilon|_{\Omega} = \cdots\ u_1\ - p = \cdots \left(\sum_{i=1}^{n} \alpha_i \phi_i \right) - p \tag{4.2.3}$$

which must be orthogonalized with respect to the same approximation function ϕk, i.e.:

$$\varepsilon, \phi_k\ = \int_{\Omega} \varepsilon \phi_k d\Omega = 0, \forall k = 1,2,\ldots,n \tag{4.2.3}$$

If the operator $\cdots$ is linear, the above equation yields a system of linear equations from which The αk coefficients can be obtained:

$$\varepsilon, \phi_k\ = \int_{\Omega} \blacklozenge\ \phi_i\quad \phi_k d\Omega = \int_{\Omega} p \phi_k d\Omega, \forall i,k\ = 1,2,\ldots,n \tag{4.2.4}$$

With

$$\langle \phi_i, \phi_k \rangle = \int_{\Omega} ◆ \phi_i \; \phi_k d\Omega, \quad \forall i,k = 1,2,\ldots,n$$

$$\langle p \; \phi_k \rangle = \int_{\Omega} p\phi_k d\Omega, \quad \forall k = 1,2,\ldots,n$$

or in the matrix form $\mathbf{A}\alpha = p$ that can be solved by a method presented in volume 1 of this work.

$$\mathbf{A}\alpha = p \rightarrow \begin{cases} \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^T \\ p = \begin{bmatrix} p \; \phi_1 & p \; \phi_2 & \cdots & p \; \phi_n \end{bmatrix}^T \\ A = \begin{bmatrix} \phi_1, \phi_1 & \phi_2, \phi_1 & \cdots & \phi_n, \phi_1 \\ \phi_1, \phi_2 & \phi_2, \phi_2 & \cdots & \phi_n, \phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1, \phi_n & \phi_2, \phi_n & \cdots & \phi_n, \phi_n \end{bmatrix} \end{cases} \tag{4.2.5}$$

5 CHOICE OF APPROXIMATION FUNCTIONS

When using MRP in any of its variants, one of the most important things is the proper choice of approximation functions. This choice gives the importance and power of the method, in which the known information of the problem is incorporated into the approximate solutions. In low-order approximations or with the use of few approximation functions, a good choice influences the result, however, in cases of high-order approximation this influence is minimized and this influence is replaced by the desired numerical convergence order. Thus, the rate of convergence becomes preponderant.

Thus, the initial step is to choose a set of approximation functions that satisfies the largest number of boundary conditions of the problem, noting that this set of functions needs to be linearly independent and complete. In general, polynomials are complete and LI, because any function can be expanded in terms of the former.

The completeness condition of the function set ensures that the solution can be expressed with a sufficient number of terms to do so.

There are two essential conditions for the proper choice of the set of approximation functions: meeting the conditions of symmetry and the conditions of contour. If the boundary conditions are of the approximation type and $z \; x,y \; = f \; x,y$

$$z \; x,y \; = f \; x,y \; + \sum_{i=1}^{n} a_i y_i \; x,y \tag{5.1.1}$$

Where $y_i = 0$ outline. These functions, in the simplest cases, can be polynomials that must obey boundary conditions and symmetry conditions. Boundary conditions that include derivatives must also be obeyed, and it is convenient to combine the residue of the differential and boundary equation. Always try to use orthogonal polynomials and their combinations to fit the boundary conditions whether they are first, second or third species, and they give computational advantages when implemented, due to their simplicity. Transcendental functions can also be used, but in general it increases the difficulty of programming and has an additional computational cost when compared to the use of polynomial functions.

In problems that are time-dependent, it is convenient to expand the solution in spatial terms so that they satisfy the boundary conditions

$$Z\ x,t\ = f\ x\ + \sum_{i=1}^{n} A_i\ t\ X_i\ x \tag{5.1.2}$$

The functions Ai t are determined by approximate methods as well as the initial conditions.

On eigenvalue problems of type

$$\begin{aligned} &\mathcal{L}u+\lambda\mathcal{N}u=0 \\ &B_k i = 0, \quad k = 1,2,\ldots,m \text{ no contorno} \end{aligned} \tag{5.1.3}$$

Where $\mathcal{L}$ $\mathcal{N}$ are generic differential operators. In most cases you have . In $\mathcal{N}$u=u l, the problem has a solution only for discrete values of the eigenvalue . In this type λ f problem the goal is to approximate the eigenvalues and eigenfunctions, thus expanding the approximation function into a series of functions, each satisfying the homogeneous boundary conditions given in (12.5.16):

$$u = \sum_{i=1}^{n} c_i u_i, \quad B_k u_i = 0 \text{ no contorno} \tag{5.1.4}$$

The approximation function above is substituted in the differential equation to form the residue, the which should be ugly orthogonal with respect to the weighting functions w_j:

$$\begin{aligned} &\sum_{i=1}^{n} [\ w_j, \mathcal{L}u_i\ + \lambda\ w_j, \mathcal{N}u_i\ ]c_i = 0 \\ &ou \\ &\sum_{i=1}^{n} [A_{ji} + \lambda B_{ji}]c_i = 0 \end{aligned} \tag{5.1.5}$$

This set of n homogeneous linear equations for the constants ci has a non-trivial solution if and only if its determinant is zero:

$$\det\; A_{ji} + \lambda B_{ji} \;\; = 0 \tag{5.1.6}$$

The above equation is a polynomial λ of degree *n and has* n roots, which are the approximations of eigenvalues. Usually these roots are distinct and real. Due to the equivalence to variational methods, Galerkin's method is preferred to be applied and because under certain conditions the eigenvalues are stationary or not sensitive to errors in the approximation of the eigenfunctions.

Example

Be the problem

$$y'' + \lambda\; 1 - x^2\;\; y = 0 \quad \therefore y\;\; 0\;\; = y\;\; 1\;\; = 0 \tag{5.1.7}$$

This problem is complicated by the 1-x2 factor; Without it the exact solution is known and is given by

$$y_i = \sin\left[\frac{1}{2}\;\; 2i - 1\;\; \pi x\right], \quad \lambda_i = \frac{1}{4}\;\; 2i - 1\;^{2}\; \pi^2 \tag{5.1.8}$$

The above functions (solving the problem without the term 1 – x2) meets the boundary conditions and provides a good source of approximation functions; polynomials are also a good source such as

$$y = c_1 \sin\left(\frac{\pi x}{2}\right) \tag{5.1.9}$$

$$\rightarrow y'' = -\left(\frac{\pi}{2}\right)^2 c_1 \sin\left(\frac{\pi x}{2}\right)$$

Soon the residue will be given by

$$\rightarrow \varepsilon\;\; y\;\; = -\left(\frac{\pi}{2}\right)^2 c_1 \sin\left(\frac{\pi x}{2}\right) + \lambda\;\; 1 - x^2\;\; c_1 \sin\left(\frac{\pi x}{2}\right) \tag{5.1.10}$$

By making the orthogonal residue the weighting function (which in the case of Galerkin's method, are the approximation functions $\sin \frac{\pi x}{2}$ themselves) using the properties of the scalar product, we have

$$
\begin{aligned}
\left(\varepsilon(y), \sin\left(\frac{\pi x}{2}\right)\right) &= \int_0^1 \varepsilon(y) \sin\left(\frac{\pi x}{2}\right) dx = 0 \\
&= \left(\frac{\pi x}{2}\right)^2 \int_0^1 \sin^2\left(\frac{\pi x}{2}\right) dx - \lambda \int_0^1 (1 - x^2) \sin^2\left(\frac{\pi x}{2}\right) dx = 0 \\
&\rightarrow \lambda = \frac{\left(\frac{\pi x}{2}\right)^2 \int_0^1 \sin^2\left(\frac{\pi x}{2}\right) dx}{\int_0^1 (1 - x^2) \sin^2\left(\frac{\pi x}{2}\right) dx} = 5.317
\end{aligned}
\tag{5.1.11}
$$

In a second approximation using (5.1.6) we have

$$
\begin{aligned}
A_{ji} &= -(2j-1)^2 \left(\frac{\pi}{2}\right)^2 \int_0^1 \sin\left((2i-1)\frac{\pi x}{2}\right) \sin\left((2j-1)\frac{\pi x}{2}\right) dx = \\
&= -(2j-1)^2 \frac{\pi^2}{8} \delta_{ij} \\
B_{ji} &= \int_0^1 (1-x^2) \sin\left((2i-1)\frac{\pi x}{2}\right) \sin\left((2j-1)\frac{\pi x}{2}\right) dx \\
&= \begin{cases} \dfrac{1}{2} - \dfrac{1}{\pi^2 (2j-1)^2}, & i = j \\[2ex] \dfrac{(-1)^{i+j+1}\left[1/(i-j)^2 + 1/(i+j-1)^2\right]}{\pi^2}, & i \neq j \end{cases}
\end{aligned}
\tag{5.1.12}
$$

Writing a program in Matlab to solve the problem:

```
% ======================================================================
% ========     PROBLEMA DE AUTO VALOR Y"+\LAMBA(1-x^2)Y=0      ==========
% ========     Y=c(I)SIN(PI.x/2)                               ==========
% ========     PELO MRP - GALERKIN: (W,Lu)= 0                  ==========
% ========        det(A(j,i) + \lambdaB(j,i)) = 0              ==========
% ======================================================================
clearvars; clc;
N = input('Entre com N número de lambdas a achar:');
if N <= 1
    N = 2;
end
disp('Problema de Autovalor');
A = zeros(N,N);
B = zeros(N,N);
for i = 1:N
    for j = 1:N
        if i == j
            deltaij = 1;
            B(j,i) = (1./3) - 1. / ((pi^2*(2*j-1)^2));
        else
            deltaij = 0;
            parc1  = 1. / (i-j)^2;
            parc2  = 1. / (i+j-1)^2;
            parc12 = parc1 + parc2;
            parc3  = (-1.)^(j+i+1);
            B(j,i) = parc3*parc12/pi^2;
        end
        A(j,i) = -((2*j-1)^2*(pi^2/8))*deltaij;
    end
```

```
end
syms lambda
delta = det(A+lambda*B);
S = solve(delta);
raizes = vpa(S);
for k = 1:N
    fprintf('\n Lambda %i = %f',k,raizes(k));
end
```

Running the above program for N=2, 3, 4 and 5, for example, we get the following results

For λi roots:

5.1253	45.5428	0	0	0
5.1222	39.6799	136.700	0	
5.1218	39.6711	106.3614	296.5355	0
5.1217	39.6644	106.2795	206.4257	544.6093

These are approximate values; The actual values for the first 3 eigenvalues are:

$$\lambda_1 = 5.122; \quad \lambda_2 = 39.66; \quad \lambda_3 = 106.3$$

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