The problem of the 1-center in graphs: Variations and applications in the resizing of electric energy

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ABSTRACT
Localization problems have applications in several areas, including the study of the planning of power distribution networks. In the present work, we present the problem of the modified 1-center in trees with applications to the study of the resizing of energy networks, as well as an algorithm for the resolution of the problem in time O(n log n), where we consider positive weights and distances.
The work also includes the presentation of computational results for some of the methods presented, as well as new strategies for the application of localization problems to the design of power distribution networks.

Keywords: Localization problems, 1-center, Centroid.

1 INTRODUCTION

The distribution of electricity in Brazil has regulations and technical standards that must be followed, aiming at excellence in the provision of the service [ANEEL, 2015]. According to [Garcia, 2003], the electric power distribution system is the part of the power system that covers from the lowering substations to the transformers (primary distribution system) and from these to the electrical input of consumers (secondary distribution system).

Considering factors that impede the control of the State in planning or relocating new distribution networks in less favored regions, in many cases, a percentage of the population takes the initiative in the acquisition of public services, even if provisionally. In the case of electricity, sometimes its supply occurs through non-regular connections, with the redirection of energy from the main electrical system to homes and/or businesses initially not registered.

In order to serve the population in such regions in a standard way and, on the other hand, representing a benefit to electric power concessionaires, the study and elaboration of a new energy network became imperative, as well as the study of the problem of resizing electricity. Certain applications of graph partitioning and localization algorithms to the planning and resizing of power grids are presented in [Assis et al., 2014], [Garcia, 2003] and [Silva et al., 1996].

In this work, we present the modified 1-center problem, applied to the re-dimensioning of a tree power grid, as well as an O(n *log* n) time algorithm for its solution where we consider positive weights and distances. The work includes a first study of the problem for the general class of graphs and computational results obtained in the implementation of the method.

1.1 LOCALIZATION PROBLEMS IN GRAPHS

The 1-median *problem* in a weighted graph looks for a vertex that minimizes the sum of the weighted distances from the vertex itself to all other vertices, each associated with a given positive weight. [Hakimi, 1964] presented a study on the 1-median problem, and presented the concepts of absolute center and absolute median as generalizations of the center and median concepts of a weighted graph as well as procedures for obtaining such locations. As a well-solved special case, the problem of the 1-median in a tree was first considered by [Hua, 1961] (Hua *et al.* Applications of mathematical models to wheat harvesting. *Chin. Math.* 2, 77-91 (1962)). An efficient algorithm for this problem is also presented by [Goldman, 1971]. [Burkard and Krarup, 1998] presented a method for solving the 1-median problem in cacti (i.e., graphs in which every block is a simple cycle or a single edge) when negative weights are allowed.

The *p-median* problem in a graph is the problem of identifying a subjoint *P of p* vertices that minimize the sum of the weighted distances from each other vertex in the graph to the nearest vertex in *P* . By [Kariv and Hakimi, 1979b], the p- median problem in a graph is NP-Hard. The problem remains NP-difficult even when restricted to planar graphs with the maximum degree equal to 3. However, in [Kariv and Hakimi, 1979b] results are presented that led to efficient algorithms when the graph is an a tree, including an algorithm that finds the p-median of an a'tree (for $p > 1$) in time $O(n^2p^2)$. [Tamir, 1996] presented an algorithm in time $O(pn^2)$ for the problem, and [Das- kin and Maass, 2015] presented some existing results in the literature for the classical problem.

The 1-center *problem* can be defined as follows: given a set of *n* demand points, a space of viable locations for a service, and a function to calculate the cost of transportation between a service and any point of demand, find a position of the service that minimizes the maximum cost of transportation from the service to the point of demand. The problem of the weighted absolute 1-center was defined and solved by [Ha- kimi, 1964]. For the absolute 1-center problem, it is possible that its solution lies at a vertex of the graph or at a point inside an edge (other than the end of such an edge). [Hakimi et al., 1978] showed that the method of [Hakimi, 1964] can be implemented so as to require O(|E|n2 log *n*). Later refinements of the procedure were obtained by [Kariv and Hakimi, 1979b], resulting in an O(|*E*|*n* log *n*) for the weighted case and $O(|E|n)$ for the unweighted case. An algorithm for obtaining the absolute 1-center of a tree is also presented by [Dearing and Francis, 1974], by [Kariv and Hakimi, 1979a] and by [Megiddo, 1983].

Given a set $X = \{x_1, ..., x_p\}$ *of* p points in a graph, the distance $d(X, vj)$ between X and a node vj is computed as $min_{i=1,...,p}\, d(x_i, v_j)$. In the *p-center problem, we must find a set* X of p points in the graph such that $\max_{j=1,...,n}\; wj.d(X, v_j)$ is minimi- zed, where wj is the weight of vj in the graph. The p-center problem was formulated by [Hakimi, 1965]. [Kariv and Hakimi, 1979a] showed that the problem in a general graph is NP-Hard. Additionally, they described an $O(n^2 \log n)$ time algorithm to obtain the absolute p-center of a vertex-weighted tree. [Tansel et al., 1983] presented a *survey* on the problems of the *p-median and the* p-center. [Wang and Zhang, 2021] present a time-rhythm algorithm $O(n \log n)$ for the *p-center problem in a tree.*

Some variations of localization problems are addressed by [Gørtz and Wirth, 2006] and [Nguyen et al., 2019] and [Calik et al., 2015]. [Gørtz and Wirth, 2006] address variations of the p-center problem, and present an approximation algorithm for the asymmetric version of the problem. In *the inverse 1-center problem* in a graph, we must modify the compressions of the edges or the weights of the vertices within certain limits, so that the pre-specified vertex becomes a 1-center 1 (absolute) of the perturbed graph and the cost of modification is minimized. [Nguyen et al., 2019] present a study on the inverse 1-center problem in a cost-weighted tree of edge length modification.

2 THE MODIFIED 1-CENTER PROBLEM

In the study of the resizing of energy networks, we aimed to define an optimal point in the network that meets all the constraints presented in loss forms, involving intrinsic constants of materials and equipment employed. Taking into account the relationship between the resistance of an electrical conductor and its length, aiming at minimizing maximum losses, algorithms for minmax localization problems can be applied to the problem of resizing power grids. We will present below the problem of the modified 1-center, as well as the concept of accumulated demands, applied to a previously partitioned network in a tree, according to an upper limit for the sets of demands in a region.

Let $T = (V, E)$ *be a weighted tree*, with a weight function $k(v_i)$ associating with each vertex $v_i \in V$ a positive real, and a f distance d*(*em*)* *associating with each edge* in ∈ *And a positive real*. Being *in* $= (v_i, v_j)$, we will denote by $d(em) = d(v_i, v_j)$ the distance between v_i and v_j.

Let $V' \subseteq V$. The weight function k can be extended to the vertex set V' *as the sum of the weights of the vertices in* V. That is, we have

$$k(V') = \sum_{v_i \in V'} k(v_i).$$

Let $T' = (V', E')$ be a subtree of $T = (V, E)$. We denote by $k(T') = k(V')$ *the* weight of the subaŕvore T'. For an edge (v_i, v_j) of $T = (V, E)$, *let* $T(v_i)$ be the tree T rooted in vi and let $T(v_i, v_j)$ be the subtree of $T(v_i)$ rooted in v_j.

In *the problem of network resizing*, we consider a secondary energy network, given by a weighted tree $T = (V, E)$. We consider a fixed source of energy t to be implanted in a vertex of the tree, called transformer. In the problem, we must define the location of a transformer, which minimizes the voltage drop by the most in the network. We observed that the length of an energy conductor has a strong influence on the voltage drop resulting from the increase in electrical resistance.

For the tree $T = (V, E)$, if the transformer is installed at a vertex $v_i \in V(T)$, *we must calculate* the total voltage drop *for each vertex* $v_j \in T$*, with* $v_j \neq v_i$. The *total voltage drop* from *vi* to a vertex v_j is defined as the sum of the voltage drops occurred at each of the edges of the one path from v_i to v_j in T.

For the definition of *the voltage drop* at an edge $(v_j, v_k) \in E(T)$, we will consider the weight k(vi) *as the energy demand at vertex* $v_i \in T$. Thus, we define the *stress drop* on an edge $(v_j, v_k) \in E(T)$, denoted by $q(v_j, v_k)$, by

$$q(v_j, v_k) = \frac{d(v_j, v_k) * k(T(v_j, v_k)) * \mu(v_j, v_k)}{100}.$$

where

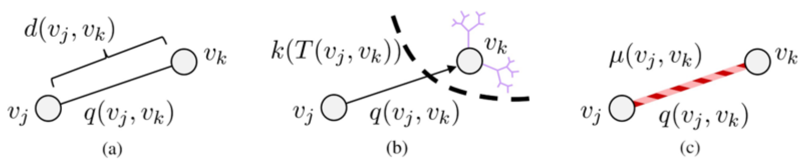
$d(v_j, v_k)$ is the distance from v_j to v_k,

$k(T(v_j, v_k))$ is the weight of the subtree $T(v_j, v_k)$, *or* cumulative demand of $T(v_j, v_k)$,

$\mu(v_j, v_k)$ = is the constant of the conductive material used in the stretch between v_j and v_k.

As illustrated in Figure 1 below:

Figure 1: Definition of a voltage drop calculation. (a) Distance between vertices, (b) Cumulative demand, (c) Material constant



We observe that for an edge $(v_j, v_k) \in E(T)$, we have $d(v_j, v_k) = d(v_k, v_j)$ and $\mu(v_j, v_k) = \mu(v_k, v_j)$. However, we do not necessarily have the symmetry $k(T(v_j, v_k)) = k(T(v_k, v_j)))$.

Let $P(v_i, v_j) = (v_i, u_1, u_2, \ldots, u_p, v_j)$ be the path in T of the vertex vi *i*n which the transformer is located to a vertex vj any of T. We define the *total voltage drop* between v_i and v_j by

$$Q(v_i, v_j) = q(v_i, u_1) + q(u_1, u_2) + \cdots + q(u_p, v_j) = \sum_{(v_k, v_l) \in P(v_i, v_j)} q(v_k, v_l).$$

That is, $Q(v_i, v_j)$ is the sum of the voltage drops occurring along the path $P(v_i, v_j)$.

For $v_i \in V(T)$, let

$$F(v_i) = \max_{v_k \in V(G)} \{Q(v_i, v_k)\}$$

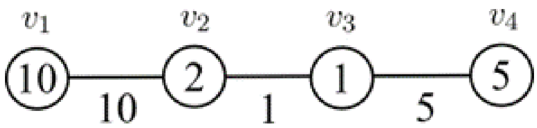
The *maximum total voltage drop from* v_i, *that is, the maximum total voltage drop considering a transformer in* v_i.

Let $v* \in V(G)$ be a vertex that minimizes this maximum voltage drop, i.e., a vertex such that

$$F(v^*) = \min_{v \in V(T)} \{F(v)\}.$$

Analogous to the terminology presented by [Kariv and Hakimi, 1979a], we define v^* *as the modified 1-center of T* and $F(v^*)$ *as the modified* 1-radius *of* T, denoted by $r1$. We observe that in the example in Figure 2, the tree T has the vertex $v1$ as 1-center and the vertex v_2 as its modified 1-center, evidencing the distinction between the two problems.

Figure 2: Example of the modified 1-center problem.



As an immediate first approach, a first method for obtaining the modified 1-center could consist of *applying n* depth searches to the T-tree , each having a vertex $v_i \in V(T)$ as the root. For each vertex $v_i \in V(T)$, the depth search rooted in vi calculates the stress drops occurring at the edges of each root-leaf path of the depth tree. Then, it calculates the total voltage drop in each leaf of the depth tree rooted *in vi* and defines the total voltage drop that has occurred, considering the transformer in *vi*. Since the ca'lculo of the accumulated demands of each suba ́rvore may require O(n) time, the method uses a pre-processing in $O(n)$ *time* for the cálculo of all the accumulated demands of the climb the tree of T. The preprocessing algorithm proceeds similarly to the algorithm of [Hua, 1961] for the 1-median in a 'tree, initiating the definition of the weights of the subtrees from the leaves of T. At the end, the modified 1-center of T will be the vertex *vi* with the lowest total tense drop at most. In view of the application of n in-depth searches on T, this first approach could require $O(n^2)$ time.

Let T be the input tree for the problem of network resizing, with weights at vertices and distances at strictly positive edges. Let v be a vertex of T with degree dv. Let $T - v$ *be* the graph obtained from T by removing the vertex v. Aiming at an adequacy to the results presented by [Kariv and Hakimi, 1979a], $T - v$ consists of *dv subtrees* $T_{v,1}, T_{v,2}, \ldots, T_{v,dv}$. We denote by $T^+_{v,i}$ the subtree consisting of $T_{v,i}$, the vertex *v*, *and the edge that connects v to* $T_{v,i}$. Based on the results presented by [Kariv and Hakimi, 1979a], we have the following extension.

Motto 1. *Let* $v \in V(G)$ be a fixed vertex and let $\tilde{v}$ be a *vertex such that the total voltage drop is greater than v, i.e.* $Q(v, \tilde{v}) = max_{v' \in V} Q(v, v')$. *Let* $T_{v,l}$ be the subtree of $T -\ v$ *to which* $\tilde{v}$ *belongs. Then, the modified 1-center of T is in* $T^{+}_{v,l}$.

Demonstration. Assume that the modified 1-center v^* of T is not in $T^{+}_{v,l}$. Then $Q(v^*, \tilde{v}) > Q(v, \tilde{v})$, assuming that the demands and distances are all positive in T. Therefore, if r_1 *is the modified 1*-radius of T, *then* $r_1 \geq Q(v^*, \tilde{v}) > Q(v, \tilde{v}) = max_{v' \in V} Q(v, v')$. Thus, by an argument analogous to that presented in [Kariv and Hakimi, 1979a], the choice of *v as the modified 1-center* of T is better than the choice of v^*, which is a contradiction.

2.1 THE MOTTO LEADS TO THE FOLLOWING RESULT.

Corolla 1. *Let* $\tilde{v}$ *and* $\hat{v}$ *be two vertices such that* $\tilde{v} \in T_{v,l}$, $\hat{v} \in T_{v,k}$, $k \neq l$, *and* $Q(v, \tilde{v}) = Q(v, \hat{v}) = max_{v' \in V} Q(v, v')$. *Then v is the modified 1-center of* T.

Analogous to an observation in [Kariv and Hakimi, 1979a], if the vertex *v* in lemma 1 is not a leaf of T, *then* $T^{+}_{v,l}$ is a proper subtree of T.

3 METHODS

Analogous to that presented by [Kariv and Hakimi, 1979a] for the 1-center problem, we can obtain the modified 1-center of a T-weighted tree , succinctly applying Lemma 1, limiting each time the search space for the vertex with the optimal location. That is, for a given vertex v_i, we calculate all possible paths from it, keeping the value that represents the drop of tension that most often occurred. By the above lemma, the climb of $T(v_i)$ in which the voltage drop occurs from vi will contain the modified 1-center of T.

For a choice of the initial vertex v_i, we will present the strategy due to [Kariv and Hakimi, 1979a] that will enable a reduction of the search time.

3.1 THE CENTROID METHOD

Let *T be a tree*, $v \in V(T)$, and let $T_{v,1}, T_{v,2}, \ldots, T_{v,dv}$ be the subtrees of $T -\ v$. Be $|T|$ the number of vertices in T and define $N(v)$ by:

$$N(v) = \max_{1 \leq i \leq dv} \{|TV,I|\}.$$

By [Kariv and Hakimi, 1979a], a *center of* the tree is a vertex v_c *for* which $N(v)$ is minimal. That is

$$N(v_c) = \min_{v \in V} \{N(v)\}.$$

We observe that a tree can have either a centroid or two. In the latter case, the two centroids are connected by an edge (see [Harary, 1969]). Additionally, we observed that $N(v_c) \leq \lfloor n/2 \rfloor$. More exactly, the number of vertices in each of the subtrees $T_{v_c,1}, T_{v_c,2}, \ldots, T_{v_c,d_{v_c}}$ *is no greater than* $\lfloor n/2 \rfloor + 1$.

Thus, if in the method for obtaining the modified 1-center, at each step we choose a centroid of *Ti* to be the vertex *vi*, then $|T_{i+1}| \leq \lfloor |T_i|/2 \rfloor + 1$, halving the number of vertices of the current tree at each step. Let be the number of vertices of the trees at each step given by $O(n), O(\frac{n}{2}), O(\frac{n}{4}), \ldots, O(1)$, *the number of steps will be in the* máximo $O(\log n)$. Therefore, analogous to the strategy presented by [Kariv and Hakimi, 1979a], for obtaining the modified 1-center of T in time $O(n \log n)$, *we must provide* an O(n) *time* algorithm for the definition of the centroid of an a'tree.

By [Kariv and Hakimi, 1979a], the inequality $N(v_c) \leq \lfloor n/2 \rfloor$ is a necessary and sufficient condition for a vertex *to* be a centroid of the tree. Thus, based on this property, a version of the algorithm of [Goldman, 1971] can be used to define the centroid of a tree in $O(n)$ steps. By [Kariv and Hakimi, 1979a], in the execution the algorithm we use a copy *of the original* T' *tree* as *an auxiliary tree* on which the algorithm will run. A variable $n(v)$ is also used for each vertex v of the tree. By treating T' during the algorithm, we can verify that if v is a sheet of T', then $T' - v$ is contained in one of the subtrees of $T - v$ and $n(v)$ then provides the number of vertices of this subtree

CENTROID(T)

$T' = T$ // Initialization

for each vertex *in* $\in T'$ do // Initialization

$n(v) = n - 1$;

while the auxiliary tree T' *does not consist of a single vertex* v_0 let v be a leaf of the auxiliary tree T' ;

if $n(v) \leq \lfloor n/2 \rfloor$ then

STOP // v It is a centroid of the original tree T ;

otherwise

let u be the vertex adjacent to v in T';

$n(u) = n(u) - (n - n(v))$;

Remove the vertex v (*and the edge* (u, v)) of T';

Return the vertex v_0 as a centroid of T ;

By [Kariv and Hakimi, 1979a], a detailed demonstration of the validity of the centroid algorithm is not presented, or a formal demonstration of its $O(n)$ time complexity. (By an observation in [Kariv and Hakimi, 1979a], on the conditional of the algorithm, if $n(v)$ *is the number of vertices of the subtree of* $T - v$ to *which* you *belong, so* $n - n(v)$ *is the number of vertices of the subtree of* $T - u$ *containing* v. Therefore, $n(u)$ is correctly updated, *i.e.* When *you* becomes a leaf, $n(u)$ gives the number of vertices of the subtree of $T - u$ containing $T' - u$. Additionally, if the relation $n(v) \leq \lfloor n/2 \rfloor$ *is not valid, then* v cannot be a centroid and its removal of T' still preserves the centroid(s) of T *in* T' (see [Kariv and Hakimi, 1979a].)

4 ALGORITHM FOR THE MODIFIED 1-CENTER PROBLEM

Analogous to the approach by [Kariv and Hakimi, 1979a], with the use of the C ENTROID method, we can obtain the modified 1-center of a tree in time O(n *log* n). In

The following algorithm, the variables T', T'', *and* T''' *represent at each step the subtrees* T_i, $T^{+}_{v_i,l_i}$, and $T_{v_{i,li}}$, respectively.

1-MODIFIED CENTER(T)

$T' = T$

while T' has more than one edge do

vc = CENTROIDE(T') ;

let $\tilde{v} \in T$ *such* that $Q(v_c, \tilde{v}) = max_{v' \in V(T)}\ Q(v_c, v')$;

let T''' *be the component* of $T - v_c$ containing $\tilde{v}$;

let T'' be the subtree consisting of T''', the vertex v_c,

and the edge that connects v_c to T''';

if there exists a vertex v *such that* $\bar{v} \notin T''$ *and* $Q(vc, \bar{v}) = Q(v_c, \tilde{v})$ then

Return the vertex v_c;

By the Corolla 1, the vertex vc is the modified 1-center of T;

$T' = T' \cap T''$;

if T' has a single edge *and* = (v_r, v_s) then

let dr = max $_{v' \in V}$ $Q(v_r, v')$;

let d_s = max$_{v' \in V}$ $Q(v_s, v')$;

if $d_r < d_s$ then Rreturn the vertex v_r ;

or else return the vertex v_s ;

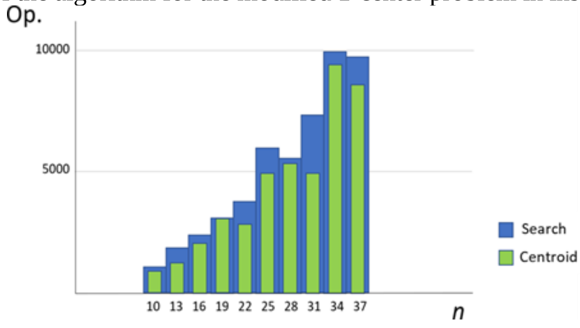
By Lemma 1, at each iteration given *you* ∈ T', the modified 1-center of T is located in T''. As a halting condition of the algorithm, we have either obtaining the modified 1-center v_c, or the reduction of T' to a single edge containing the vertex to be returned.

Since the number of iterations is limited by $O(\log n)$, and considering the time O(n) required Centroid, the *1*-Center Modified method requires O(n log *n) time*.

5 FINDINGS

Localization problems have applications that include covering a particular region with the location of *hospitals*, mobile phone towers, installing warning sirens, and the problem of locating battery depots in a drone delivery network (see [Liu, 2019]), among others. In this section, we present a first evaluation of the computational performance of the method for the modified 1-center, such performance can be seen in Figure 3. The experiments were performed using an Intel(R) Core(TM) i7 - 9750H CPU @ 2.60GHz, 16.0 GB RAM - Windows 11.

Figure 3: Execution of the algorithm for the modified 1-center problem in instances (n = 10, . . . , 37).



The development in C++ of the software used included the implementation of classes for the storage of instances of the problem in linear space, and for the execution of the CENTROID methods in *O(n)* time and 1-CENTER M*ODIFIED* in O(n log *n)* time. The approach included the creation of instâˆncias of sizes of 10, 13, 16, ..., and 37 vertices, as well as a performance comparison with the first method of measuring the 1-center modified in quadratic time. For the set of generated instâˆncias, with the representation of weights of vertices and edges, accumulated demands, voltage drops and electrical conductivity constants in simple precision, the modified 1-CENTRO METHOD represented an improvement of 18.92% on average, in the execution time in elementary operations).

As new strategies for solving the problem of resizing energy networks, we can mention the application of a variation of the problem of the inverse 1-center, aiming at an adaptation of sectors of a network in project to the capacity of equipment used in the distribution of energy.

6 CONCLUSION

We present a method for solving the modified 1-center problem in trees in O(n *log* n*)* time . The method used the centroid approach presented by [Kariv and Hakimi, 1979a] for the problem of the 1-cla´ssic center in the 'trees. For graphs without isthmus, the concept of accumulated demands may represent an equivalence of the problem presented to the classical problem, requiring an in-depth study of the problem when restricted to other classes of graphs. The computational results presented show that the *O*(n *log* n*)* method represents a significant improvement in execution time when compared to initial quadratic approaches to the problem.

REFERENCES

ANEEL, A. N. d. E. E. (2015). Web page. https://antigo.aneel.gov.br/web/ guest/regulacao-dos-servicos-de-distribuicao. Acessado: 2022-06-06.

Assis, L., Franca, P., e Usberti, F. (2014). A redistricting problem applied to meter readingin power distribution networks. *Computers & Operations Research*, 41:65–75.

Burkard, R. e Krarup, J. (1998). A linear algorithm for the pos/neg-weighted 1-median pro- blem on a cactus. *Computing*, 60:193–215.

Calik, H., Labbé, M., e Yaman, H. (2015). *p*-center problems, g. laporte, s. nickel, f. saldanha da gama (eds.). In Springer, editor, *Location Science*. Springer International Publishing Switzerland.

Daskin, M. S. e Maass, K. L. (2015). The p-median problem, g. laporte, s. nickel, f. saldanha da gama (eds.). In Springer, editor, *Location Science*. Springer International Publishing Switzerland.

Dearing, P. M. e Francis, R. L. (1974). A minimax location problem on a network. *Transpor- tation Sci.*, 8:333–343.

Garcia, V. (2003). Grasp para o problema de planejamento de redes secundárias de distribuição de energia elétrica. In *Anais do XXXV SBPO*, p. 1427–1437, Rio de Janeiro. SOBRAPO.

Goldman, A. J. (1971). Optimal center location in simple networks. *Transportation Sci.*, 5: 212–221.

Gørtz, I. L. e Wirth, A. (2006). Asymmetry in k-center variants. *Theoret. Comput. Sci*, 361: 188–199.

Hakimi, S. L. (1965). Optimum distribution of switching centers in a communication network and some related graph theoretic problems. *Operations Research*, 13:462–475.

Hakimi, S. L., Schmeichel, E. F., e Pierce, J. G. (1978). On *p*-centers in networks. *Transporta- tion Sci.*, 12:1–15.

Hakimi, S. (1964). Optimal locations of switching centers and the absolute centers and medi- ans of a graph. *Operations Research*, 12:450–459.

Harary, F. (1969). *Graph Theory*. Addison-Wesley, Reading, MA.

Hua, L. K. (1961). Applications of mathematical models to wheat harvesting (english transla- tion in *chin. math.* 2 (1962), 77-91). *Acta Mathematica Sinica*, 11:63–75.

Kariv, O. e Hakimi, S. (1979a). An algorithmic approach to network location problems. part 1: The *p*-centers. *SIAM J. Appl. Math.*, 37:513–538.

Kariv, O. e Hakimi, S. (1979b). An algorithmic approach to network location problems. part 2: The *p*-medians. *SIAM J. Appl. Math.*, 37:539–560.

Liu, Y. (2019). An optimization-driven dynamic vehicle routing algorithm for on-demand meal delivery using drones. *Computers & Operations Research*, 111:1–20.

Megiddo, N. (1983). Linear-time algorithms for linear programming in r^3 and related pro- blems. *SIAM J. Comput.*, 12:759–776.

Nguyen, K. T., Nguyen-Thu, H., e Hung, N. T. (2019). Combinatorial algorithms for the uniform-cost inverse 1-center problem on weighted trees. *Acta Math. Vietnam.*, 44:813–831.

Silva, M., Franc¸a, P., e Silveira, P. (1996). Long-range planning of power distribution systems: Secundary networks. *Computers Elect. Engng*, 22:179–191.

Tamir, A. (1996). An $o(pn^2)$ algorithm for the p-median and related problems on tree graphs. *Oper. Res. Lett.*, 19:59–64.

Tansel, B., Francis, R., e Lowe, T. (1983). Location on networks: A survey. part i: The p-center and p-median problems. *Management Science*, 29:482–497.

Wang, H. e Zhang, J. (2021). An $o(nlogn)$-time algorithm for the k-center problem in trees. *SIAM J. Comput.*, 50:602–635.