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The differences between the definitions of trapezoid in mathematics textbooks in Brazil

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ABSTRACT
In geometry books and mathematics textbooks, two types of definition for trapezoids are found, one in which parallelograms are considered trapezoids and another in which they are not, therefore, these configurations determine two different hierarchical classifications for quadrilaterals. Which of these definitions is used most frequently in Brazilian mathematics textbooks? What problems can be caused by this duplicity of definition types? These questions drove the realization of this research, which consists of an analysis of mathematics textbooks for the 6th year of Elementary School. Of the twenty-five titles analyzed, most have definitions that do not consider parallelograms as trapezoids. It was possible to observe that each book presents the definition of trapezoid as if it were unique, which can create an obstacle for the student's learning. A joint study between mathematicians and mathematics educators is considered important to deepen the theme and analyze the cognitive cost of maintaining different definitions for trapezoids in geometry.

Keywords: Trapezoid, Textbook, Geometry.

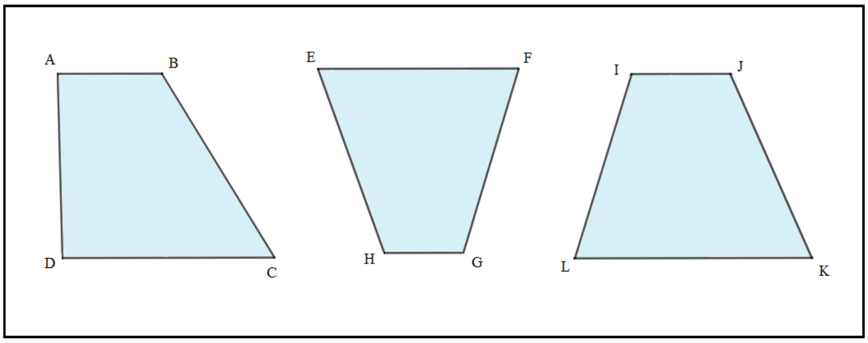
1 INTRODUCTION

Geometry comprises the study of a vast set of concepts and procedures essential for solving problems in the physical world and in different areas of knowledge. According to the National Common Curricular Base – BNCC (BRAZIL, 2017), in basic education should be developed the skills to recognize and compare polygons such as triangles, squares, rectangles, trapezoids and parallelograms and classify them in relation to their sides and vertices. This document points out that Mathematics is not restricted only to the quantification of deterministic phenomena and the techniques of calculation with numbers, it creates abstract systems, which organize and interrelate phenomena of space, movement, shapes and numbers. These systems contain ideas and objects that are fundamental to understanding phenomena, constructing meaningful representations, and consistent argumentations in varied contexts.

The BNCC (BRAZIL, 2017) highlights that, in the 6th grade of elementary school, the ability to identify characteristics of the quadrilaterals, classify them in relation to sides and angles and recognize the inclusion and intersection of classes between them must be developed.

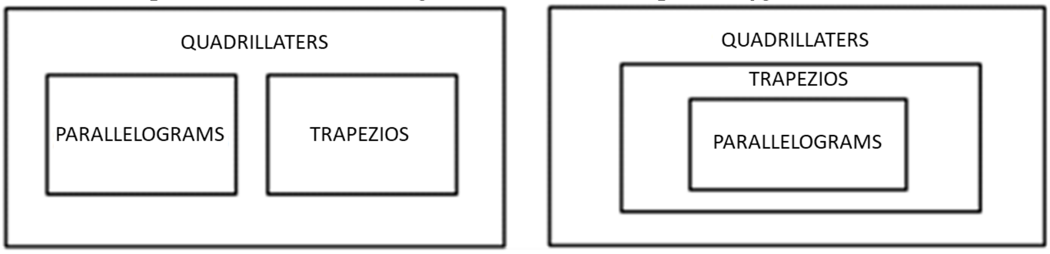
The trapezoid is a four-sided quadrilateral with two parallel opposite sides and is usually represented in the books as shown in Figure 1.

Figure 1: Trapezoid representations.



The question is: "trapezoid is a quadrilateral that has a pair of parallel sides" or "trapezoid is a quadrilateral that has only one pair of parallel sides ". This small difference in the way of defining this geometric object generates a considerable difference in the classification of the quadrilaterals. The first definition: "trapezoid is a quadrilateral that has a pair of parallel sides" allows one to consider parallelograms as trapezoids, since the parallelogram has two pairs of parallel sides, therefore it has a pair, as this definition indicates. The definition: "trapezoid is a quadrilateral that has only one pair of parallel sides" excludes the possibility of a parallelogram being considered trapezius. These definitions determine two different hierarchical classifications for the quadrilaterals (figure 2), and in geometry textbooks and mathematics textbooks, these two types of definition for trapezoid are found.

Figure 2: Classification of quadrilaterals according to the type of definition



Which of these definitions is used most often in Brazilian mathematics textbooks? What problems can be caused by this duplication of definition types? These questions drove the realization of this research, which consists of an analysis of mathematics textbooks, of the sixth grade of elementary school, and seeks to understand how the definition of trapezoid and its characterization are treated in these books.

Raymond Duval's Theory of Records of Semiotic Representation and Guy Brousseau's notion of epistemological obstacle underpinned the data analysis.

2 THEORETICAL BASIS

Duval (1995) states that the cognitive activities involved in the learning of mathematics require the use of systems of expression and representation that go beyond natural language and images. In the case of geometry, geometric figures, statements, perspective representations and symbolic notations stand out. For Duval (2011), mathematics is the only domain in which the advancement of knowledge is closely linked to the creation of new semiotic systems, which give access to mathematical objects. For the author, geometric figures differ from other visual representations by the fact that there are several ways to recognize their forms or figural units, that is, to see mathematically a figure it is necessary to change the look without the representation being modified.

According to Duval (1999), some semiotic representations, such as drawings, aim to be iconic representations, that is, there is a relationship between the representation and the object represented, so that the recognition is immediate, without the need for additional information, for example a tree, a car, a house, etc. In mathematics, visualization does not work with iconic representations, looking at a record of representation is not enough to understand what is actually represented. For the author, seeing a figure in geometry is a much more complex cognitive activity than the simple recognition of an image, it is necessary the discriminative recognition of the forms and the identification of the objects that correspond to the recognized forms. And that requires specific training. Visualization in mathematics is necessary because it presents the organization of relations, but it is not instinctive, for it is not mere visual perception.

Duval (1988) states that, in geometry, to understand a geometric object, in addition to the immediate visual recognition of the form, it is necessary to explain mathematical properties and articulate the figure with discursive elements, identifying properties of the figure that are not visually explicit. According to Duval (2005), geometry, among the fields of knowledge that students must acquire, is what requires the most complete cognitive activity, because it mobilizes the gesture, the language and the look, being necessary to build, see and reason. It is also a difficult area to teach, because, regardless of the goals, the results achieved have been disappointing.

The Curricular Proposal for the education of young people and adults (BRASIL, 2002), points out that the contents of geometry are not developed with due attention, although they contribute decisively to the development of intellectual capacities such as spatial perception, creativity, hypothetical-deductive reasoning, in addition to allowing various relationships between Mathematics and other areas. According to this document, to be taught, mathematical knowledge must be transformed and it is up to the teacher, in addition to having the necessary knowledge for this, to understand the obstacles involved in the process of construction of concepts and procedures, as well as other aspects related to student learning.

The notion of epistemological obstacle was developed by Bachelard (1996) and points to the idea that error does not mean failure but an important step to advance in new knowledge.

Brousseau (1983) applied the concept of epistemological obstacle to the Didactics of Mathematics. For him, an obstacle is a set of difficulties related to a knowledge that has been adapted to a specific case or in special conditions, but when a new situation arises and the need for ruptures, this knowledge becomes an obstacle, because the individual resists the novelties in defense of the already established knowledge. According to the author, obstacles are not only manifested through errors, but also through the impossibility of facing certain problems or solving them properly.

Brousseau (1983) defines four types of obstacles, corresponding to different ways in which they can be treated in the didactics of mathematics. These four types of obstacles are classified into:

- Epistemological, linked to conceptual difficulties, that is, it stems from the lack of in-depth knowledge of the content or understanding of its process of historical development.
- Didactic, arise from the choice of teaching methods to be used by the teacher. They may arise, for example, from the oversimplification of a complex abstract concept (Topaz Effect) or from the abusive use of metaphors (Jourdain Effect) as a single way of understanding a concept.
- Psychological, linked to the affective nature of the subject or to the beliefs of the community in which he is inserted.
- Ontogenetic, when the subject is not mature enough, or when the difficulties of his psychogenetic development prevent him from understanding a new concept.

Could the fact that a geometric object has definitions that generate different interpretations aggravate the problems and generate obstacles to the teaching and learning of geometry? This question refers to the double interpretation in relation to the definition of trapezoid in geometry. After all, is the parallelogram a trapezoid or not?

Euclid (300 BC), in definition 22 of book 1 of his work "The Elements", presents the following classification for quadrilateral figures:

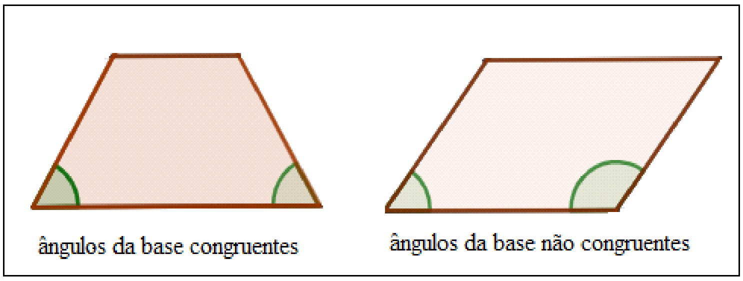
> ... on the one hand, square is that which is both equilateral and rectangular, and, on the other hand, oblong, that which, on the one hand, is rectangular, and, on the other hand, is not equilateral, while rhombus, which, on the one hand, is equilateral, and, on the other hand, is not rectangular, and rhomboid, which has both opposite sides and equal opposite angles to each other, which is neither equilateral nor rectangular; and the quadrilaterals, in addition to these, are called trapezoids (Euclid, 2009, p.98).

He calls oblong what we call rectangle and rhomboid what we call parallelogram, non-rectangle and non-equilateral. According to this classification, Euclid defines trapezoid as a figure that does not fit the previous definitions, that is, that is neither square, nor rectangle, nor rhombus, nor parallelogram.

Garbi (2010) calls as trapezoid the quadrilateral that has two parallel and different sides. Machado (2012) defines trapezoid as a quadrilateral that has a pair of opposite sides parallel to each other, and the other two sides not parallel to each other. Both definitions exclude the possibility of a parallelogram being considered trapezius. Papa Neto (2017) defines trapezoid as a convex quadrilateral that has two of its parallel sides and clarifies that if the two parallel sides are congruent, the trapezoid in question is a parallelogram.

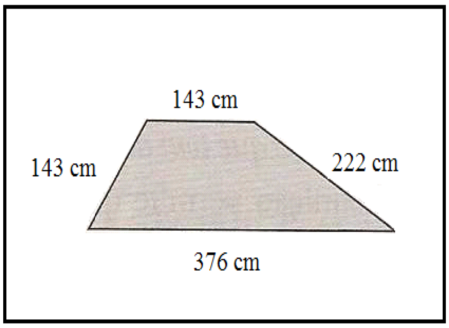
According to Bongiovanni (2010), when considering the definition that admits parallelograms as being trapezoids, it is necessary to be careful with the definition of isosceles trapezium. The definition of isosceles trapezoid as being a trapezoid that has congruent non-parallel opposite sides, which is very common in textbooks, ceases to be consistent because parallelograms do not have non-parallel opposite sides. The definition "isosceles trapezoid is the trapezoid that has at least one pair of congruent opposite sides", which includes all parallelograms as isosceles trapezoids, also has a drawback, as it invalidates the particularity that the angles of the base of an isosceles trapezoid are congruent (figure 3), which is a practice incorporated by students and teachers.

Figure 3: Angles of the base of an isosceles trapezoid and a parallelogram.



The author points out that, if a change is made in the definition so that it is "isosceles trapezoid is a trapezoid that has a single pair of congruent sides", to exclude parallelograms, there is an inconsistency, because it is possible the existence of trapezoids that have two congruent sides that are not isosceles, as shown in Figure 4.

Figure 4: Trapezoid with two congruent and non-isosceles sides.



Source: Bongiovanni, 2010.

According to Bongiovanni (2010), a consistent definition would be: Isosceles trapezoid is the trapezoid that has a single pair of congruent opposite sides, which would exclude all parallelograms.

Ferreira and Almouloud (2017) analyzed geometry books indicated in undergraduate courses in mathematics and describe that, in the three books analyzed, the definitions of trapezoid and isosceles trapezoid adopted are inconsistent with the stated properties, and point to the need to make a restriction on the definition of isosceles trapezoid to fit the results considered.

Maioli (2002) organized a workshop for teacher training, focusing on quadrilaterals and reports that teachers stated that to accept the rectangle as a trapezoid it would be necessary to change some concepts already rooted. He also mentions the case of a teacher who stated that she would continue to represent the trapezoid in a traditional way, because she was still trying to accept that the rectangle could be a trapeze.

Studies addressing the different definitions of trapezoid have also been conducted in other countries. Popovic (2012) states that two different definitions of a trapezoid are commonly found in mathematics textbooks in the United States and that, depending on the definition assumed, the answer to a given task may be different, with the possibility that the student's answer may be considered wrong if he is not using the same definition used by the teacher. Perrin-Glorian (1998) asks: "How to define an isosceles trapezoid?" and discusses the possible contradictions in this definition when we consider parallelograms as trapezoids. Gouthier (2019) makes a study on the classification of quadrilaterals, with emphasis on the case of trapezoids. Díaz and Fernández (2011) discuss the problem that occurs when teachers have to make decisions about which definition and which classification of convex quadrilaterals to adopt in their pedagogical practices.

The results of these researches corroborate the importance of discussion and reflection in relation to the choice or even the unification of the definition of the trapezoid geometric object.

3 METHODOLOGY

In order to support the reflection on the definition of trapezoid in geometry, an analysis of mathematics textbooks was carried out to investigate how these books approach this subject. The textbook is a material widely used in Brazilian schools, especially because it is distributed free of charge in public schools. He assists teachers in conducting their pedagogical practice, hence the importance of knowing how this material treats the subject in question.

Twenty-five titles of mathematics textbooks of the 6th grade of elementary school - final years, with date of edition between 1981 and 2018, were selected for analysis. Among these books are the eleven titles indicated in the Digital Guide of the National Textbook Program (PNLD) of 2020 of Mathematics. This program evaluates and makes available didactic, pedagogical and literary works free of charge to Brazilian public schools of basic education. The works that make up this guide are approved in pedagogical evaluations coordinated by the Ministry of Education, which has the participation of specific technical commissions, integrated by specialists from different areas of knowledge. Thus, the books that are part of this program are those that are (or should be) used in Brazilian public schools.

Although the books that are part of the PNLD represent the recent works used in Brazilian public schools, it was decided to also analyze works that are not part of this plan, even with older dates, to have parameters of comparison in relation to the definition used in these publications, because they present a content that may have been taught previously, including individuals who are currently teachers.

In these works, the ways of defining the trapezoid geometric object and classifying it among the quadrilaterals were analyzed, as well as the approach of the proposed exercises.

In this research there is a combination of quantitative and qualitative approaches, so the method can be characterized as mixed. According to Creswell (2010), the strategy used was sequential explanatory, which is characterized by the analysis and collection of quantitative data in a first phase and collection and analysis of qualitative data in a second phase, which is based on the analysis of quantitative data.

4 RESULTS AND ANALYSIS

The definitions of trapezoid found in textbooks were divided into two groups:

- G1: Definitions that do not consider parallelograms as trapezoids.
- G2: Definitions that consider parallelograms as trapezoids.

For ease of viewing, a table was created relating each heading to one of the trapezoid definition groups. The books are in ascending order of release date. From item 15, the titles that are part of the Digital Guide of the PNLD 2020, Mathematics curricular component, appear. In cases where more than one edition of the same title was analyzed, the works appear as a subitem of the table.

Table 1: Textbooks analyzed.

No.	TITLE MATHEMATICS TEXTBOOK - 6th YEAR	G1	G2
1	Mathematics: 5th grade (SARDELLA E MATTA, 1981)	x	
2	Mathematics: concepts and stories (NETTO, 1996)	x	
3	Mathematics an adventure of thought (GUELLI, 2001)		x
4	a) Mathematics in the right measure (JAKUBOVIC, LELLIS and CENTURIÓN, 2001) b) Mathematics nowadays: in the right measure (CENTURIÓN and JAKUBOVIC, 2015)	x	
5	Everything is Mathematics (DANTE, 2007)	x	
6	Learning Mathematics (GIOVANNI and PARENTE, 2007)	x	
7	Discovering and applying Mathematics (MAZZIEIRO, 2012)	x	
8	Mathematics (IMENES and LELLIS, 2012)		x
9	Practicing Mathematics (ANDRINI and VASCONCELLOS, 2012)		x
10	Languages and applications: Mathematics (YOUSSEF, PACHI and HESSEL, 2015)	x	
11	Mathematics of everyday life (BIGODE, 2015)	x	
12	Mathematics: ideas and challenges (MORI and ONAGA, 2015)	x	

13	Mathematics: connection point (LOPES, ALENCAR and ALENCAR, 2015)	x	
14	Mathematics and reality (IEZZI, DOLCE and MACHADO, 2018)		x
15	a) The conquest of mathematics (GIOVANNI, CASTRUCCI and GIOVANNI JR, 1998)	x	
	b) The conquest of mathematics (GIOVANNI JR and CASTRUCCI, 2009)		
	c) The conquest of mathematics (GIOVANNI JR and CASTRUCCI, 2018) (PNLD 2020)		
16	Apoema: Mathematics (LONGEN, 2018) (PNLD 2020)		x
17	Araribá Mais: Matemática (GAY and SILVA, 2018) (PNLD 2020)	x	
18	Mathematical Convergences (CHAVANTE, 2018) (PNLD 2020)	x	
19	Generation Alpha Mathematics (OLIVEIRA and FUGITA, 2018) (PNLD 2020)	x	
20	Mathematics- Bianchini (BIANCHINI, 2018) (PNLD 2020)	x	
21	Mathematics: comprehension and practice (SILVEIRA, 2018) (PNLD 2020)	x	
22	Essential mathematics (PATARO and BALESTRI, 2018) (PNLD 2020)	x	
23	Mathematics reality & technology (SOUZA, 2018) (PNLD 2020)	x	
24	Telaris Mathematics (DANTE, 2018) (PNLD 2020)	x	
25	Tracks of Mathematics (SAMPAIO, 2018) (PNLD 2020)	x	

Table 2 quantitatively summarizes the data presented in table 1.

Table 2: Number of titers for each trapezoid definition group

TEXTBOOKS	**G1**	**G2**
Non-PNLD 2020 titles	10	4
Titles belonging to the PNLD 2020	10	1
Total	**20**	**5**

In the books of G1 seven ways of enunciating the definition of trapezoid were observed, they are similar, but present particularities in the syntax that may be of interest, so it was decided to present them separately:

- "When only two opposite sides are parallel, the quadrilateral is called a trapezium" (item 1 of the table).
- "Trapezoids are quadrilaterals that have two parallel sides and two non-parallel sides" (items 2, 6, 15a, 15b of the table).
- "Trapezoid is every quadrilateral that has only two parallel sides" (items 4a, 4b, 5, 11, 12, 15c of the table).
- "Trapezoids have a single pair of parallel sides." (item 7 of the table).
- "Trapezoid is a quadrilateral that has exactly one pair of parallel sides" (item 13 of the table).
- "Quadrilateral that has only one pair of parallel sides" (items 12, 17, 18, 19, 20, 21, 22, 23, 24 of the table).
- "A quadrilateral that has only one pair of parallel opposite sides is called a trapezium" (item 25 of the table).

In the G2 books, four ways of enunciating the definition were observed:

a. "A quadrilateral that has two parallel sides is called a trapezium" (items 3, 16).

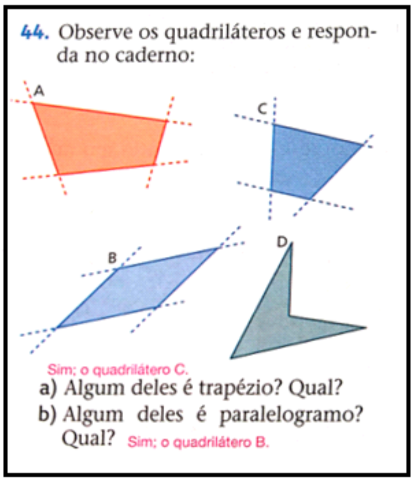
b. Trapezoids are quadrilaterals that have 1 pair of parallel sides (item 9).

c. "Trapezoid: It is a simple quadrilateral that has two parallel sides" (item 14).

d. "In trapezoids there are a pair of parallel sides that are their bases" (item 8).

The definition *d*, despite having been considered from G2, generated some doubts, because if a trapezoid has a pair of parallel sides that are its bases, if it has two pairs of parallel sides will it have 4 bases? The textbook 8 (IMENES and LELLIS, 2012) that presents this definition, brings, on page 97, an exercise in which it asks the student to identify, among four quadrilaterals, if any of them is trapezoid and, in the answer, considers trapezoid only the quadrilateral that has only two parallel sides, disregarding a parallelogram that was also part of the set of figures to be analyzed. This can be verified in the exercise presented in Figure 5, which in the answer of item *a*, considers only the quadrilateral C as trapezium.

Figure 5: Trapezoid identification exercise.



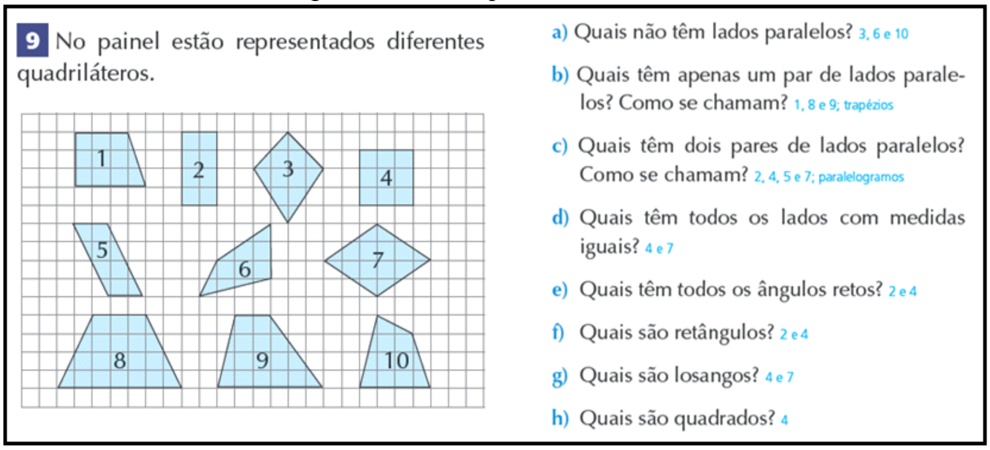
Source: Imenes and Lellis, 2012, p.97.

Thus, although this textbook presents a definition that by its syntax is considered G2, analyzing the proposed exercises, it is possible to conclude that it guides the student to consider trapezoid as being a quadrilateral with only a pair of parallel sides. This exercise presents the trapezoid geometric figure only as an iconic representation, detached from the definition previously proposed, disregarding that, in geometry, to understand a geometric object it is necessary the visual recognition articulated with discursive elements that allow to identify the properties of the figure that are not visually explicit (DUVAL 1988).

Book 16 (LONGEN, 2018, p.135) brings the definition "A quadrilateral that has two parallel sides is called a trapeze", but adds: "A trapezoid can have up to two right angles". With this statement he excludes the possibility of rectangular parallelograms being considered trapezoids. Therefore, this book brings an initial definition that makes it possible to consider parallelograms as trapezoids, but in the sequence excludes part of this possibility, and may lead the user of this textbook to conclude that parallelograms are not part of the set of trapezoids, contradicting the initial definition.

Books 3 (GUELLI, 2001), 9 (ANDRINI and VASCONCELLOS, 2012) and 14 (IEZZI, DOLCE and MACHADO, 2018) explain that they consider parallelograms to be trapezoids, because they present a pair of parallel sides. Still, book 9 presents an exercise (figure 6) in which it induces the idea that trapezoid is a quadrilateral that has only one pair of parallel sides.

Figure 6: Exercise presented in book 9.


9 No painel estão representados diferentes quadriláteros.

a) Quais não têm lados paralelos? 3, 6 e 10

b) Quais têm apenas um par de lados paralelos? Como se chamam? 1, 8 e 9; trapézios

c) Quais têm dois pares de lados paralelos? Como se chamam? 2, 4, 5 e 7; paralelogramos

d) Quais têm todos os lados com medidas iguais? 4 e 7

e) Quais têm todos os ângulos retos? 2 e 4

f) Quais são retângulos? 2 e 4

g) Quais são losangos? 4 e 7

h) Quais são quadrados? 4

Source: Andrini and Vasconcellos, 2012, p.157.

In the exercise presented in Figure 6, when the author relates trapezoid to quadrilaterals that have only one pair of parallel sides (item b) and when he calls parallelograms the quadrilaterals that have two pairs of parallel sides (item c) without making any relation to the trapezium, it weakens the idea that parallelograms can be considered trapezoids.

These contradictions, between the definition of trapezoid presented and the interpretations observed in the proposed activities, can cause obstacles to the learning of students who use these books if there is no adequate intervention on the part of the teacher. As defined by Brousseau (1986), an obstacle is a set of difficulties related to a knowledge that is adequate for a specific case, but does not adapt to a new situation, resulting in the need for ruptures with the already established knowledge. In the case of the definition of trapezoid it would not be necessary to break with the previous knowledge, because in fact the two knowledge could be considered correct, but, without this clarification, the obstacle generated can be even more paralyzing for the development of the concept by the student, because each school phase may present a different path to follow, without him being aware of which is correct, in this case, both.

Most of the books analyzed proposed exercises for the identification and classification of quadrilaterals, similar to the one shown in Figure 4. Only textbooks 3, 10, 11, 16 and 25 did not present this type of exercise and only five books classified trapezoids into isosceles, rectangles and scalenes (books 1, 10, 15, 19 and 21).

It is possible to observe that most of the books that present the definition that admits the parallelogram as being trapezoid do not make clear the relationship between the representation of the parallelogram and the definition of trapeze, making it difficult for the student to make this association. For Duval (2005), the visualization of a geometric figure is not mere visual perception, it presents the organization of relations, it is necessary the discriminative recognition of the forms and the explicitness of their properties. Thus, the lack of elucidation of this relationship in textbooks hinders the visualization of the geometric object.

If in the domain of mathematics the advancement of knowledge is intrinsically linked to the creation of semiotic systems (DUVAL, 2011), these systems must be perfectly established, without the possibility of generating indeterminacies, or doubts, because they are the ones that give access to mathematical objects, especially geometric objects.

It is important to emphasize that, regardless of the definition of trapezoid adopted, the formulas for calculating the area of the trapezoid and/or the Euler median[1] presented in books of the following years, continue to use variables that represent "major base" and "minor base", which could be considered an incongruity when classifying parallelogram as trapeze, because in this case, The bases would be the same. Thus, even if the same collection of books is used in the following grades, if the teacher is different, a different approach or interpretation of the definition of trapezoid previously used may be used.

Another problem observed is that the books present the definition of trapezoid as if it were unique, they do not bring as one of the definitions but as "the definition", causing the student to assume it as definitive and it is possible that not even the teacher has the correct information to get around the problem.

5 FINAL CONSIDERATIONS

Of the twenty-five titles analyzed, twenty present definitions that fall under G1, which do not consider parallelograms as trapezoids and five present definitions that fall under G2, which admit parallelograms as trapezoids. Of the titles that are part of the 2020 PNLD, only one presents a definition that falls under G2, but this same book brings the statement that a trapezoid can have up to two right angles, which excludes the possibility of rectangular parallelograms being considered as trapezoids.

The existence of different definitions for the same mathematical object can generate some problems for student learning. A student may have learned, in a given school year, one of the definitions of trapeze and, in the following year, the teacher assume another definition and evaluate the activity of this student according to this other definition, which could generate an obstacle to his learning. According to Brousseau (1983, apud ALMOULOUD, 2010) an obstacle is manifested by errors, and these occur, not by the lack of knowledge but by a previous knowledge that, for a while, was sufficient, but that reveals itself false or inadequate in a new or broad context. This obstacle does not disappear

[1] Euler's median is the line segment formed by the connection between the midpoints of the two diagonals of the trapezoid

with the learning of new knowledge, on the contrary, it opposes resistance to its acquisition, its understanding, delays its application, lasts in a latent state and reappears suddenly, especially in the previous context, when circumstances allow it.

For Brousseau (2008), the didactic obstacles are caused by the choices made by the teacher in the teaching process, choices that seem correct at the moment, but can generate a later difficulty. In the case of the trapezoid definition, the choice of the teacher, even if correct, can lead to a later obstacle, because there is no way to know which definition will be adopted by other teachers during the student's entire school phase.

After all, what is the cost of maintaining mathematical definitions that can generate different interpretations?

It is possible to perceive that the form of definition of the trapezoid is not an issue that affects only Brazil, so there is a need for a joint study between mathematicians and mathematics educators to reflect on which definition of trapezoid is more convenient, because, as Perrin-Gloria (1998) said, teachers need safe and consistent definitions.

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