CHAPTER 8

MONTE CARLO SIMULATION IN TRIANGULAR IRREGULAR NETWORKS

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ABSTRACT
Working with Digital Terrain Models (DTM) or Geographic Information Systems (GIS), Triangular Irregular Networks (TINs) is one of the most used ways to represent surface topology. Given this form or representation, this article investigates one initial probabilistic demonstration to quantify up to which accuracy σ (*sigma*) can be consider what is ambiguous from the topological point of view in 2D Delaunay triangularizations. To achieve it, we designed an initial demonstration that there is a maximum precision for which the network topology remains constant in a new Delaunay triangulation, at each point individually and in the Triangular Irregular Network as a whole. The methodological approach was experimental. Various mathematical experiments were carried out using the pseudorandom Monte Carlo Simulation method. First, for each point of the Network, and then for all network points for varied σ. The experiments culminate in helping solve the problem of the existence of maximum σ for which the probability of occurrence in constant triangular irregular network topology is 100%. The mathematical results gave rise to the following: Considering a TIN generated by Delaunay Triangulation, if any point of coordinates (x_i, y_i) in a triangular irregular network is disrupted (has its place altered), according to a Normal distribution $N(\mu, \sigma^2)$, then, exists a value σ_{max} (*sigma* maximum) for which the topology of the Network remains constant. For example, it was found that $\sigma_{max.1}$ of point 1 exists and is obtained by $\sigma_{max.1} = 0.30866$, and in point 2, $\sigma_{max.2} = 0.2$. The results also indicate the following for Triangulated Irregular Networks: Every two-dimensional irregular triangular Network generated by the Delaunay Triangulation has a value σ_* (*sigma* asterisk) to which the network topology remains constant. In this work, simulating the worst case of irregular triangular Network: $\sigma_* = 0.2$. Finally, it is concluded that the σ maximum for each point exists, as well as for the Network as a whole. However, results need to be tested in more extensive networks to prove (or not) if it always happened. We advance the knowledge on the Triangular Irregular Networks combining simulation techniques and network topology.

KEYWORDS: Triangular Irregular Network, Monte Carlo *Simulation*, *Topology*. *Delaunay triangulation*

1 INTRODUCTION

Geospatial sciences work with geometric and Geo-graphic Database (GDB) aspects. In geometric data acquisition, triangulation (Hegeman et al., 2014; Kas-trisios and Tsoulos, 2018; Kim et al., 2010) and/or trilateration (Cheng et al., 2004; Mazuelas et al., 2009) are important methods employed for populating these GDBs. These methods allow the generation of Triangular Irregular Networks (TIN) (Kastrisios and Tsoulos, 2018; Reuter et al., 2007) used in Geographic Information Systems (Kamel Boulos and Geraghty, 2020; Mollalo et al., 2020).

This processing of Irregular Triangle Networks for the generation and extraction of geo-metrics in GIS is influenced by the errors inherent in each vertex (Florinsky, 2002). In geodetic (Kastrisios and Tsoulos, 2018; Martínez et al., 2005; Sharp et al., 2019; Wang et al., 2000) and topographic (Florinsky, 2002; Li et al., 2006) triangulations, the distances between vertices are very large, indicating that probably the interference in triangle generation is very small due to the relationship between point accuracy and edge length. However, in a TIN with smaller edges these errors further influence the generation of the triangles, which may vary as the vertex precision is reduced. Such errors indicate the mathematical and physical rigidity of their triangles.

Considering in this context the acquisition of geometries for GIS from triangulation networks, this research aims to understand the following question: to what extent a change in the quality of adjustment, i.e. increasing or decreasing the accuracy in the points, the triangulation would still be considered stable. To attend to this research question, we develop an initial demonstration to show that, there is a maximum precision for which the network topology remains constant after a new Delaunay Triangulation. First simulated at each point initially and then on the TIN as a whole. From these initial experiments, it is possible to recognize patterns capable of generating new TINs with similar triangulation, same topology and same statistical quality as the original.

This article aims to contribute to results that quantify to what accuracy the σ (sigma) can consider what is topologically ambiguous in an Irregular Triangulated Network. We advance the knowledge about the topology of Triangulated Irregular Networks using Monte Carlo simulation techniques (Carmel et al., 2009; Gugiu and Dumitrache, 2005; Walędzik and Mańdziuk, 2018).

2 THEORETICAL FUNDAMENTATION

Theoretical content about the definition of Triangular Irregular Networks (TIN) by means of Delaunay Triangulation, topology of a TIN, and Monte Carlo simulation method are essential aspects for the understanding of this work.

2.1 THE CREATION OF AN TRIANGULAR IRREGULAR NETWORK

There are several algorithms for generating a Triangular Irregular Network (TIN) from a dot mesh. Among them, Delaunay Triangulation (Kastrisios and Tsoulos, 2018; Zeiler, 1999; Felgueiras and Goodchild, 1995; Tsai, 1993; Fernandes and Menezes, 2005) allows generating triangles as homogeneous and close to an equilateral triangle as possible, optimizing the represented surface. In addition, it is the most popular used for this conversion, and is present in virtually all Geographic Information Systems. Thus, TIN is a vector data format defined by a triangulation from a set of sample points irregularly distributed in coordinates (x,y), with respective z values, usually referring to altimetry, which allow mathematical modeling of a surface through a network.

2.2 TOPOLOGY OF A TRIANGULAR IRREGULAR NETWORK

The geometric topological vector relationship in an Triangular Irregular Network is the core of this article. According to Casanova et al (2005), topology is the part of mathematics that investigates the properties of configurations that remain invariant in transformations of rotation, translation and scale. These are spatial relations that are independent of geometry, but rather of the *elements of topological vector relationships*. These elements are generically: connectivity, adjacency, and contingency. Erciyes (2013), defines the *connectivity* or *connectedness* of a graph as follows: “*A graph is connected if there is a path between any pair of vertices v1 and v2."* The adjacency is the neighborhood information of spatial objects, where an edge determines the left and right polygon. Finally, *contingency* is information about the inclusion of a spatial object within another spatial object. Of these elements of the topological vector relationship, connectivity is essential to the present study, as far as TIN is concerned. The topology of an triangular irregular network is considered constant if, with changing the coordinates of the vertices, the triangles remain the same after a new Delaunay Triangulation.

2.3 MONTE CARLO SIMULATION

Another aspect essential to the understanding of this paper is the Monte Carlo method (Me-tropolis et al., 1953; Amar, 2006; Mark and Mordechai, 2011), named after the Monte Carlo Casino in the principality of Monaco. It is an application of inferential statistics. Amar (2006) describes some of the algorithms that have been developed to perform Monte Carlo simulations. In this paper, Monte Carlo simulation is used in experimentation on the effect of random errors on each coordinate of an original point in the Triangular Irregular Network.

3 METHOD

In the first stage the initial mathematical modeling is defined. The two-dimensional point being

$$A = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

In which:

$$(\forall x_1 \in x)(\exists y_1 \in y)\big((x_1, y_1) \in \Re^2\big)$$

Your errors being σ_{x_1} e σ_{y_1} associated with the coordinates in x and y; and its tendencies τ_{x_i} e τ_{y_i} associated to the coordinates in x e y. Mathematically, it starts from the premise of uncertainty associated with the geometric coordinates of each point in a Geographic Database. Therefore, analogously, also associated to line and polygon geometries. From this statement, the true Cartesian coordinate of a point A can be defined by:

$$\hat{A} = \begin{bmatrix} x_1 + \tau_x \pm \sigma_{x_1} \\ y_1 + \tau_y \pm \sigma_{y_1} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} \pm \begin{bmatrix} \sigma_{x_1} \\ \sigma_{y_1} \end{bmatrix}$$

So generically the uncertainty of a two-dimensional point can be written as:

$$\hat{P}_i = \begin{bmatrix} x_i + \tau_x \pm \sigma_{x_i} \\ y_i + \tau_y \pm \sigma_{y_i} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} \pm \begin{bmatrix} \sigma_{x_i} \\ \sigma_{y_i} \end{bmatrix}$$

As this research does not aim to study tendencies, we considered $\tau = 0$.

From these initial definitions, we considered some sets of randomized experiments with disturbance (coordinate changes) of each point in the network, called local experiments or simulations. For the purposes of this text, the terms disturbance, noise and point oscillation are used synonymously. The term simulation refers to the triangular irregular network whose points are being disturbed. Furthermore, from the experiments we sought to recognize whether there is and to what accuracy σ (*sigma*) the topology produced by the triangulation can be considered constant. With this, an algorithm was implemented to compare topologies of triangular irregular networks.

FIG. 1 Example of topology simulation of an triangular irregular network, triangulated by the Delaunay method, by changing points V1 and V4.

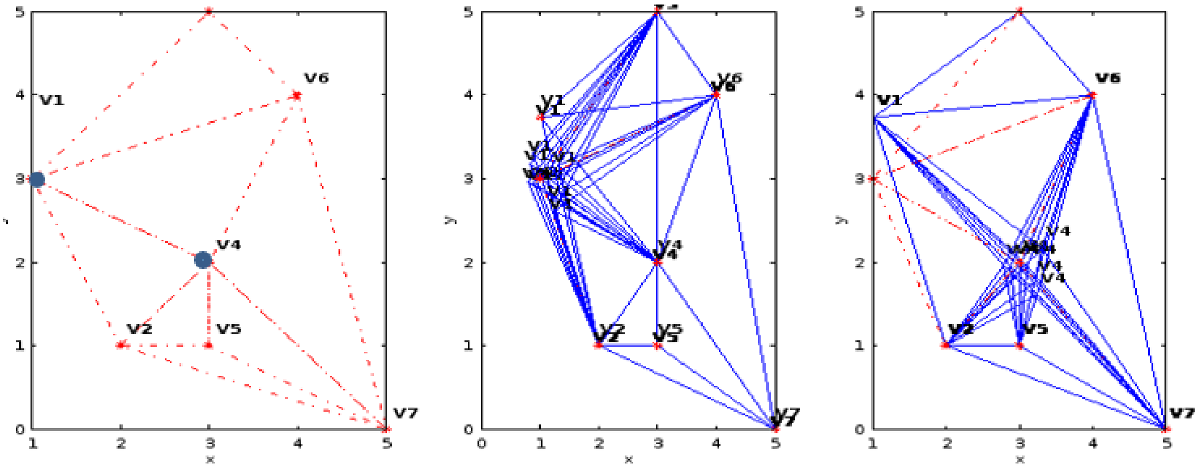


Figure 1 presents two simulation examples with constant topology. The first graph shows the original Delaunay Triangulation, in red; the second graph shows the disturbance at point 1 (V1), with the simulated triangulations in blue; and the third graph shows the disturbances at point 4 (V4), with the simulated triangulations also in blue.

The experiments were performed on a simulated network with 4 points and a lozenge configuration, in order to allow testing the methodology and the local behaviors for, in later stages of the research, applying it to triangular irregular networks with more vertices. Random mathematical experiments were performed using the Monte Carlo method.

For the simulation, at each point a noise was inserted k times, in such a way as to simulate the disturbance of the original point. The point disturbance was performed by generating noise according to the standardized normal distribution, which has mean zero and variance equal to 1, therefore:

$$z_i = \frac{x_i - \mu}{\sigma}$$

The formula was rewritten in x and y by:

$$z_{x_i} = \frac{x_i - x(ponto)}{\sigma} \text{ e } z_{y_i} = \frac{y_i - y(ponto)}{\sigma}$$

In the simulations it was done:

$$\sigma.z = \hat{x} - x(ponto) \text{ e } \sigma.z = \hat{y} - y(ponto)$$

Therefore,

$$\hat{x}_i = (\sigma . z_i) + x(ponto); \hat{y}_i = (\sigma . z_i) + y(ponto)$$

In which $\hat{x}_i$ e $\hat{y}_i$ are the new coordinates obtained by perturbing each original point and $i = 1, \ldots, k$. For the generation of normally distributed pseudo-random numbers the Marsaglia and Tsang Ziggurat method was chosen, described in Marsaglia and Tsang, 2000.

Initially, oscillations were performed for each point in the network in an isolated manner. Each vertex was initially disturbed with $\sigma_{x_i} = \sigma_{y_i} = 1$, generically modeled on the form:

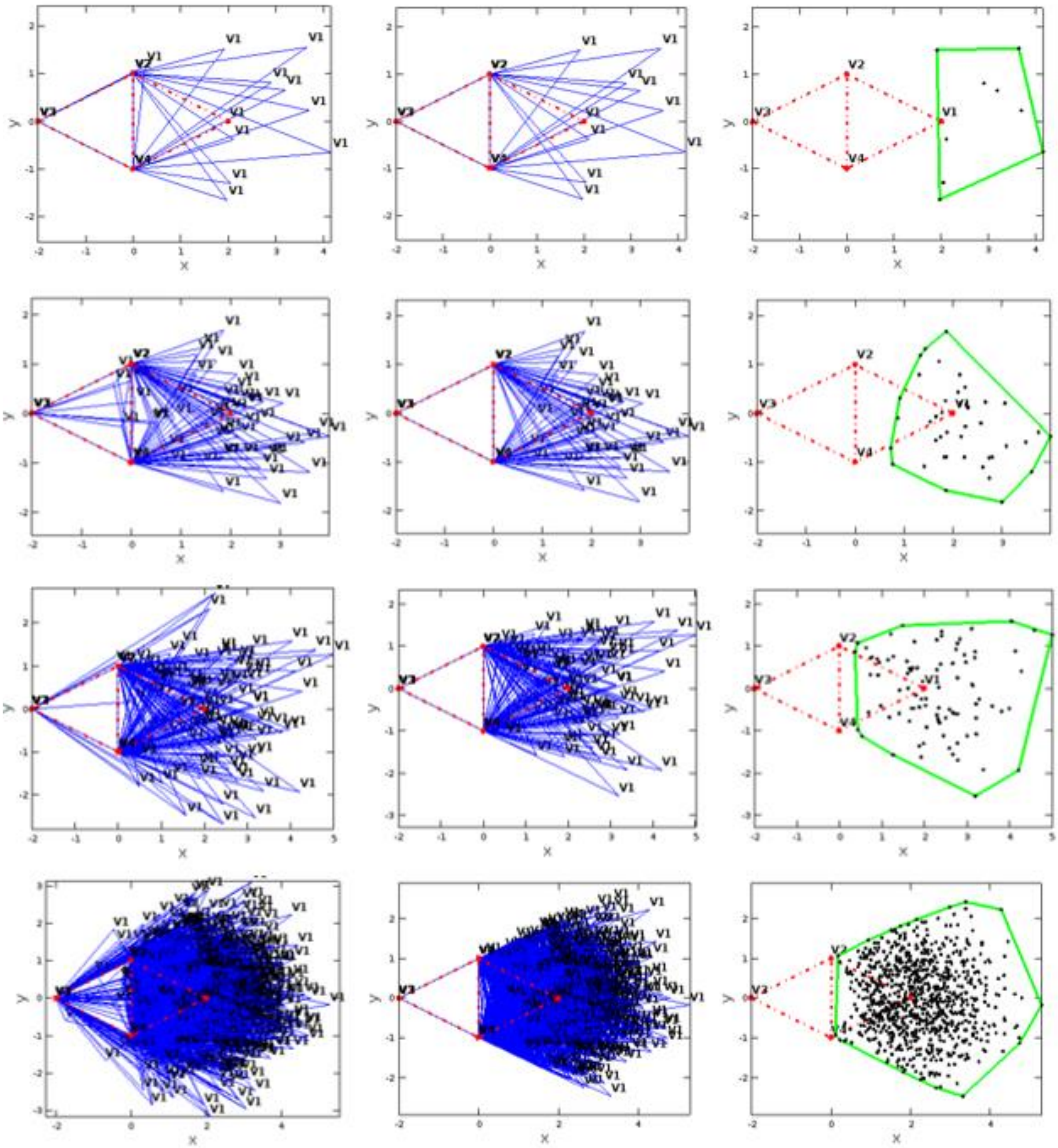
$$\hat{V}_i = \begin{bmatrix} x_i \pm \sigma_{x_i} \\ y_i \pm \sigma_{y_i} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \pm \begin{bmatrix} \sigma_{x_i} \\ \sigma_{y_i} \end{bmatrix}$$

Then the simulation was performed on the network for varied σ. The first simulated network was composed of 4 points $V_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$, forming a rhombus of coordinates $V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. This network was chosen initially because it is the smallest amount of points that allows for different triangulations during its disturbance. The choice in rhombus shape was motivated by being a simple geometry for the initial analyses.

4 RESULTS

The results of the Monte Carlo simulations are presented at each point in the network separately, followed by an analysis of the results. The figures show the simulations performed at point V_1 (figura 2 e figura 3) and in point V_2 (figura 4 e figura 5). All analyses concerning the points V_3 e V_4 are analogue, since they are symmetrical.

FIG. 1 Simulated experiments for $k = 10$, 50, 100 e 1000, all with $\sigma = 1.0$.



The figure (FIG. 2) shows the representative figures of all simulated Delaunay Triangulations: (i) Without topology comparison, i.e., keeping the k disturbances referring to the original point 1. In it, the simulated triangulations are shown in blue; (ii) The triangulations with constant topology, i.e., $k_{TC} \leq k$, in which k_{TC} is the amount of disturbances with constant topology (TC), also in blue; and (iii) The resulting convex closure of the points with noise that allowed generating Delaunay Triangulations with the same topology as the original network, shown in green. Random experiments were performed for $k = 10$, 50, 100 e 1000, presented each in a row with the 3 graphs described, initially with $\sigma = 1$. It has been shown necessary to identify for other σ what is the behavior of the region through which point 1 can oscillate and yet remain with constant topology.

FIG. 2 Local convex closures of the disturbed point for simulations with $\sigma = 0.1, 0.3, 0.5$ and 1.0; each one with $k = 10, 100, 500, 1000$.

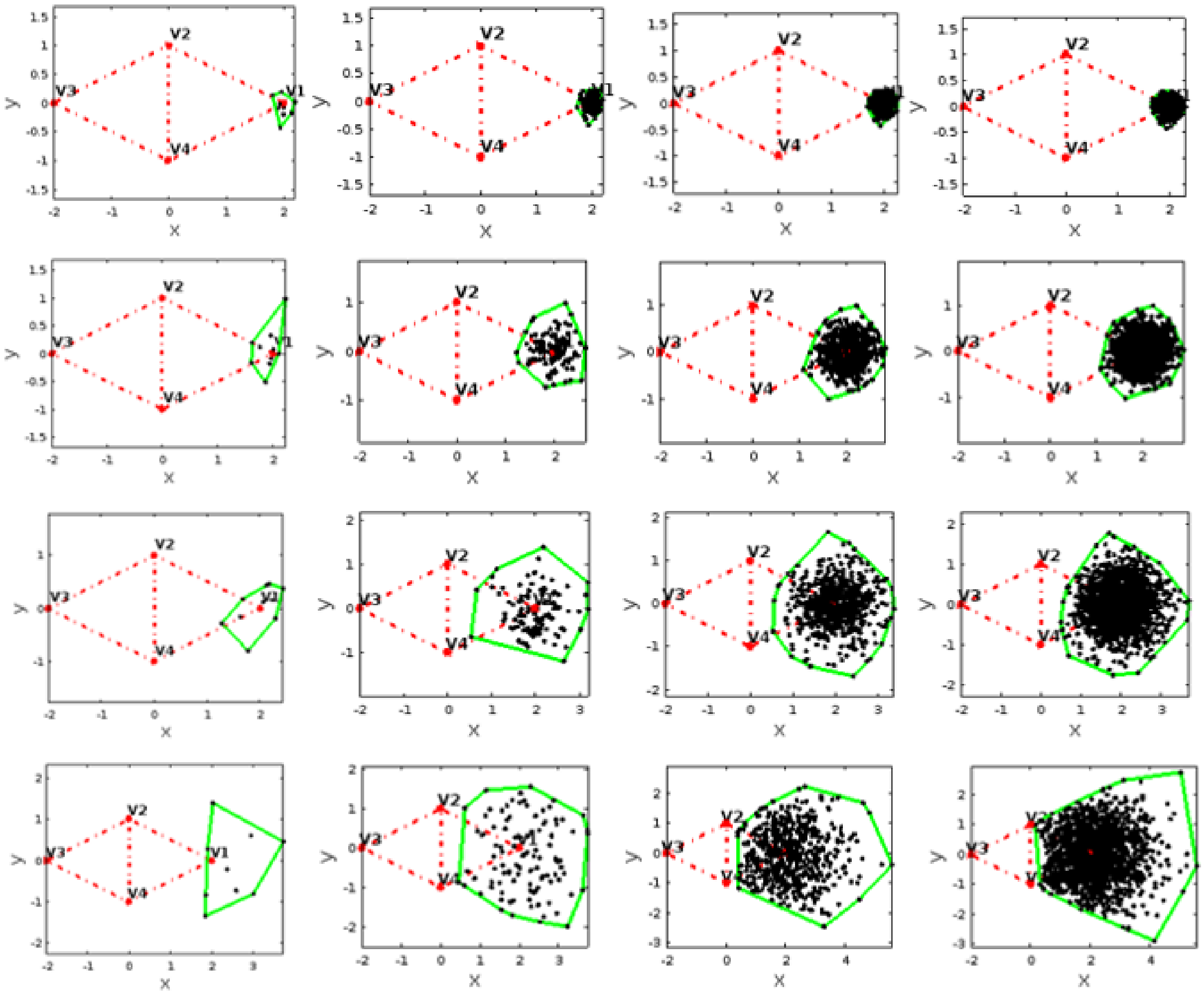
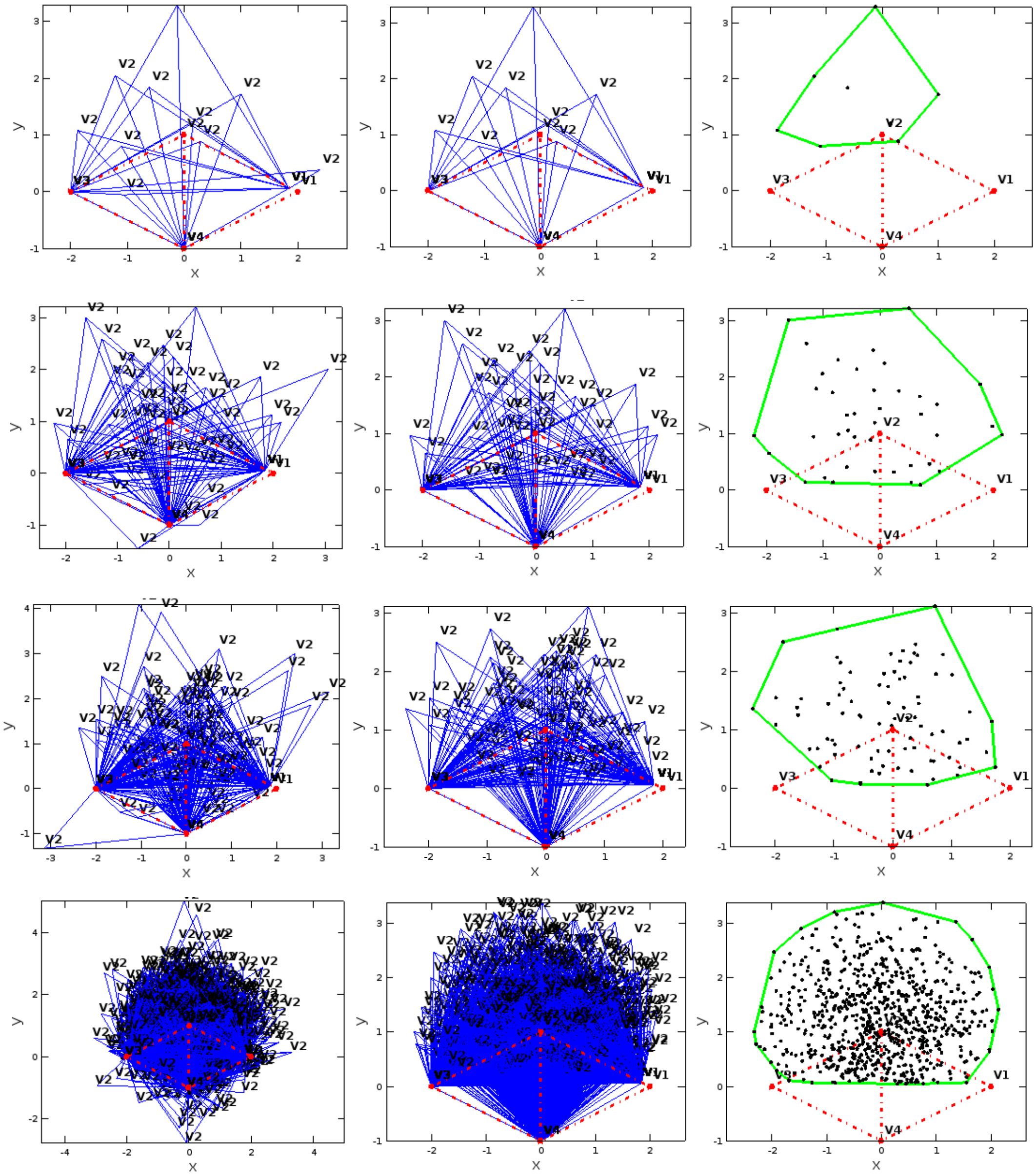


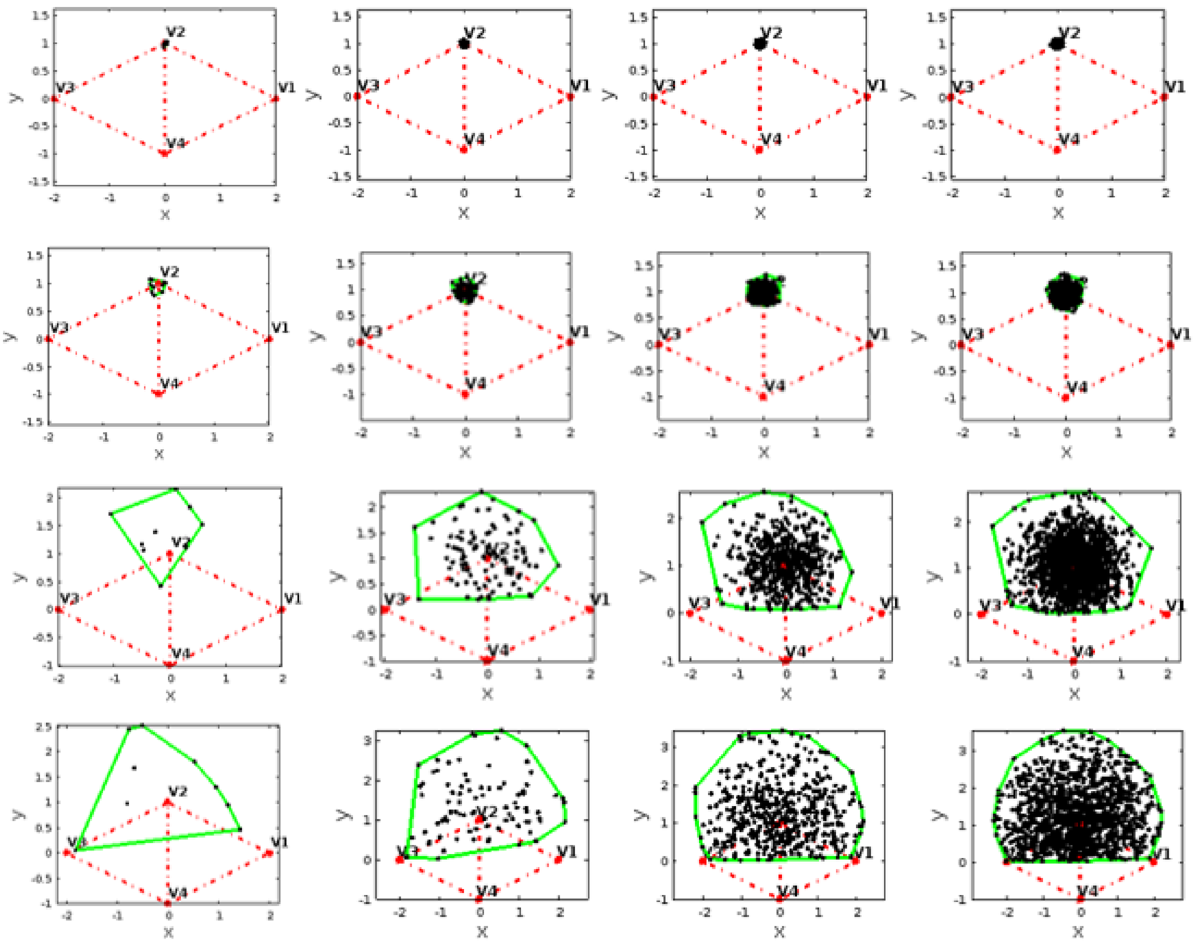
Figure 3 shows the points and convex closures that bound the constant topology region with different simulations, changing the value of σ to 0.1, 0.3, 0.5 e 1.0; each one with $k = 10$, 100, 500, 1000. In the figure are shown in each line $k = 10, 100, 500, 1000$ para $\sigma = 0.1$; $k = 10, 100, 500, 1000$ to $\sigma = 0.3$; and so forth.

FIG. 1 Simulations to k = 10, 50, 100 e 1000, with σ = 1.0.



In the second simulation we performed disturbances at the point V_2. As in vertex 1, experiments were carried out at point 2 with k = 10, 50, 100 e 1000, with σ = 1.0. Afterwards (FIG. 1), experiments allow recognizing behavior patterns when σ is variable.

FIG. 2 Experiments for the convex closure of the points with simulations for σ = 0.1, 0.3, 0.5 e 1.0; each one with k =10, 100, 500, 1000.



Likewise, in order to identify the pattern of behavior for various σ when the topology remains unchanged, simulations were performed for values of σ= 0.1, 0.3, 0.5, and 1.0; each experiment with perturbations k=10, 100, 500, 1000. Figure 5 presents these results, such that to each row of graphs are shown all experiments of a given σ, with k in increasing order.

5 DISCUSSIONS

Monte Carlo simulation is used to solve the problem of identifying the existence of maximum σ for which the probability of constant topology occurring in the irregular triangular network is 100% after a new triangulation by Delaunay's method. The mathematical results gave origin to the following statements.

Consider a Triangular Irregular Network (TIN) generated from Delaunay Triangulation. If any point of coordinates (x_i, y_i) is disturbed (has its place changed), according to a Normal distribution $N(\mu, \sigma^2)$, then there is a value σ_{max} (maximum sigma) to which the TIN topology will remain constant

Demonstration: To find the maximum sigma σ_{max} for which the topology remains constant, we performed thousands of disturbances at point 1 as follows: σ varying from 0.1 a 5 with intervals of 0.1; 0.01; 0.001; 0.0001; ...; 0.000001; according to Table 1.

TAB. 1 Simulation results with σ varying from 0.1 a 5 with intervals of 0.1; 0.01; 0.001; 0.0001; ...; 0.000001

Variation intervals of σ	**Results of $\sigma_{max.1}$ to $P(TC = 1) = 1$**	**Results of $\sigma_{min.1}$ to $P(TC = 1) < 1$**
0.1:0.01:0.5	σ = 0.38000	σ = 0.39000
0.1:0.001:0.5	σ = 0.35100	σ = 0.35200
0.1:0.0001:0.5	σ = 0.30920	σ = 0.30930
0.1:0.00001:0.5	σ = 0.30866	σ = 0.30867
0.1:0.000001:0.5	σ = 0.30909	σ = 0.30909

With this, it is verified that $\sigma_{max.1}$ of point 1 exists and is obtained by $\sigma_{max.1} = \min(\sigma_{max}) = 0.30866$. For point 2, in order to find $\sigma_{max.2}$ the simulations were performed as shown in Table 2.

TAB. 2 Simulation results with σ varying from 0.1 a 5 with intervals of 0.1; 0.01; 0.001; 0.0001; ...; 0.000001

Variation intervals of σ	**Results of $\sigma_{max.2}$ to $P(TC = 1) = 1$**	**Results of $\sigma_{min.2}$ to $P(TC = 1) < 1$**
0.1:0.1:0.5	$\sigma =$ 0.20000	$\sigma =$ 0.30000
0.1:0.01:0.5	$\sigma =$ 0.26000	$\sigma =$ 0.27000
0.1:0.001:0.5	$\sigma =$ 0.25700	$\sigma =$ 0.25800
0.1:0.0001:0.5	$\sigma =$ 0.25190	$\sigma =$ 0.25200
0.2:0.00001:0.5	$\sigma =$ 0.20240	$\sigma =$ 0.20241

With this, it was identified that $\sigma_{max.2} = 0.2$.

6 CONCLUSIONS

From the analysis of the simulated experiments performed using Monte Carlo Simulation, the following can be proposed regarding the maximum oscillation of the points in the Triangular Irregular Network.

Every Triangular Irregular Network (TIN) generated from two-dimensional Delaunay Triangulation has a value σ_* (sigma asterisk) for which the network topology will remain constant, obtained by:

$$\sigma_* = \min(max\sigma_i) = \min(\sigma_{max.1}, \sigma_{max.2}, \dots, \sigma_{max.n})$$

Demonstration: Generalizing, making the worst case triangular irregular network as: $\sigma_* = \min(\sigma_{max.1}, \sigma_{max.2}) = \min(0.30866, 0.2) = 0.2$.

Note that, although the Monte Carlo method is computationally inefficient, being dependent on a large number of disturbances (changing the coordinates of the points) to obtain the probabilistic results, it proved to be adequate for this type of demonstrative research.

Finally, it is possible to conclude, for the simulated cases, that there is a maximum σ for each point of the irregular triangular network triangulated by the Delaunay criterion, as well as, there is a maximum σ for which the network as a whole remains with its topology constant. It is suggested that in future work, simulated experiments with multiple irregular triangular networks with multiple sizes be performed, in addition to simulations on real networks to prove (or not) the statements proposed in this paper.

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