Elastic *versus* plastic bending capacity of Timoshenko-Ehrenfest beams

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ABSTRACT
The Timoshenko-Ehrenfest beam theory is widely applied in structural analysis to simplify mathematical modeling when the beam can be considered as a long structural element. In the literature, the yield strength of the material is the limit state in most applications of this structural element. However, in some situations, this design limit state can be improved to the plastic limit, increasing the structural capacity to support loads. Therefore, in this article, the elastic and plastic bending capacity of the Timoshenko-Ehrenfest beam is presented in terms of its concepts and formulation. This theory was applied to a rectangular beam with a simply supported cross-section subjected to a point load. In this specific example, the results show that when the design limit state is changed from the yield limit to the plastic limit, there is a 50% increase in the bending capacity of the beam. Based on the comparison between the two limit states carried out here, the bending capacity of a structural system can be improved in terms of an upper limit state, allowing the designer to increase the potential of a beam-like structure.

Keywords: Elastic bending, Plastic bending, Bending capacity, Timoshenko-Ehrenfest beam.

INTRODUCTION

Massive research has been conducted in the field of plasticity theory (including boundary analysis) since the second half of the last century. Hill (1950) presents the history of Plasticity Theory contextualized on a physical-mathematical level. He highlights the elastic-perfectly plastic material, yield criteria, hardening by cold deformation, stress-strain constitutive relations, general theorems and plastic anisotropy. It also presents solutions for elastoplastic solids (bars, shells, tubes) subjected to combined stresses.

In his work, Drucker (1956) shows the theory of perfect plasticity, linear and non-linear incremental theories for materials hardened by cold working. The theorems of minimum potential energy and minimum complementary potential energy are derived for stress-strain relationships, as well as establishing absolute minimum principles.

(1959) presents a new parametric form of equilibrium equations for structures that shows their relationship with compatibility conditions, the representation of static and kinematic principles as dual linear programming problems, the existence of collapse mechanisms and the representation of compatibility relations as Kirchoff cycle conditions for an associated graph.

Picón and Cañas (1987) introduce a procedure for Limit Analysis of two-dimensional frameworks using linear programming techniques and concepts, which models a small number of variables and

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constraints. They present two examples of two-dimensional frameworks. For the first, only one proportional and monotonic load is applied; for the second example, six loads are applied externally to the frame considered.

The work produced by Soares (2006) presents a finite element technique for planar frame applications using kinematic formulation by linear programming. In addition, a CAD (Computer Aided Design) system is developed and used as a pre-processor and post-processor for the formulated limit analysis technique, which is implemented using a C++ application.

In his work, Pasquali (2008) uses an alternative to possible numerical difficulties when close to the collapse load, which consists of asymptotically simulating the elastoplastic behavior using a non-linear elastic relationship. Several examples of structures using resistance criteria, such as von Mises-Hencky and Drucker-Prager, are modeled. The results obtained are compared with existing literature. In addition, the non-linear elastic relationship is used to determine the strength domain in mediums with different levels of porosity. In the contribution by Kaveh and Jahanshahi (2008), Ant Colony Systems (ACS) are applied in order to find the collapse factor of two-dimensional frames. Three variants of these systems are developed, and their relative performances are compared in two numerical examples.

Wong (2009) specifically highlights the Stiffness Method in the field of Structural Limit Analysis. He also shows the plastic structural behavior of structures, making comparisons with elastic behavior. The plastic flow rule, elastoplastic analysis, incremental elastoplastic analysis, limit analysis via linear programming, the factors affecting plastic collapse and some design considerations on plasticity are shown. Frames, beams and other structures are solved using elastoplastic analysis and limit analysis.

In his book, Hosford (2013), in addition to other subjects, introduces the history of the theory of plasticity and dislocations, isotropic production criteria, anisotropic plasticity, the effects of strain hardening and strain rate dependence, as well as plasticity test reports.

Nonato (2021) applies the theory of plasticity and limit analysis in the context of uncertain parameters to calculate the collapse load of structures and, consequently, carry out reliability analyses.

The theory of plasticity is concerned with the production phenomena of materials. Rather than being elastic, plastic deformations are permanent, which means that if the load that caused the deformations is stopped, part of the deformation remains. The theory of plasticity is studied because some design situations can allow the engineering materials involved to reach higher levels of deformation (greater than the corresponding elastic limit) without losing their design functionality, which increases the load capacity of the system.

Historically, plasticity theory has been approached according to its underlying concepts. In other words, the theories that explain the mechanisms of plasticity drive the branch of plasticity theory considered. For example, Hosford (2013) discusses three approaches to plasticity theory:

- **Continuum theory:** this is the most widely applied. It depends on yield criteria and allows the prediction of the stress states that cause yielding and the resulting deformations. Therefore, the level of cold working under different loading conditions can be compared.
- **Crystallographic sliding mechanisms:** by understanding these mechanisms, continuous behavior is explained. This approach has been almost successful in predicting anisotropic behavior.
- **Occurrence of sliding:** concerned with the mechanism of sliding. Dislocation theory shows how crystalline materials are deformed by the sliding process. However, its weak connection with continuum theory makes it difficult to apply in many practical cases.

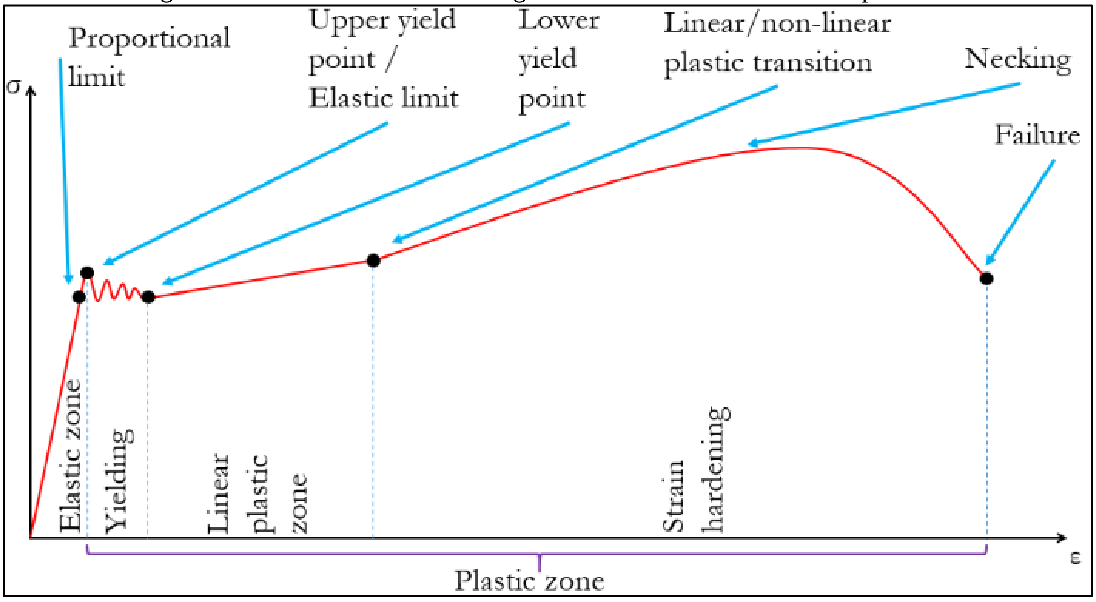
Lubliner (1990) mentions that the adjective "plastic" describes ductile materials, which have the properties of being easily formed by means of suitably applied loads and of maintaining their new geometry after the load has ceased. The author notes that considerable deformations (from plastic forming) are often accompanied by small volumetric changes. Thus, plastic deformation is fundamentally a distortion and, consequently, deviatoric stresses are largely responsible for this work.

He also states that a direct plasticity test can be carried out by producing a state of simple shear deformation in a specimen by applying a load, resulting in the aforementioned state. Therefore, when it comes to metallic materials, the test that reproduces this condition is the one in which one of the end cross-sections of a thin-walled tube is rotated around its longitudinal axis in relation to the other end (pure shear). However, it should be noted that this is not a simple test to carry out. For this reason, a tensile test is generally preferred, where the simplicity of execution and the machinery make it more attractive to check material properties.

According to Maugin (1992), plastic is the behavior of a solid in which permanent deformations occur without damage. In the context of a tensile test, the onset of plastic deformation occurs when the material of the specimen reaches its elastic limit. From then on, the specimen is subjected to load increments until it ruptures, which is the point at which plastic deformation ceases. Recovery of the elastic part of the deformation can be achieved at any time during the process, as long as the specimen is unloaded.

According to Lubliner (1990), incipient or imminent plastic collapse is a state in which a non-zero strain rate occurs under constant loading, provided that the following items are satisfied: a) all previous deformations must be of the same magnitude as the elastic deformation, so that geometric changes can be neglected; and b) acceleration can be neglected and, consequently, the problem can be treated as quasi-static. A uniaxial tensile test on a ductile material (carbon steel, for example) commonly shows the following stress-strain diagram (fig. 1):

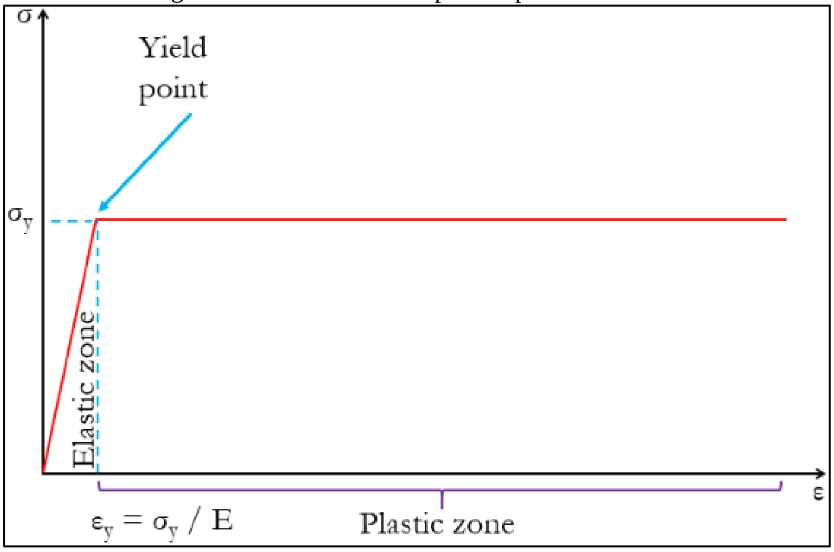
Figure 1: Idealized stress-strain diagram of a mild steel at room temperature.



Source: Own authorship, 2024.

As can be seen, the material can withstand deformations much greater than the yield strain before failure occurs, which is a measure of its ductility. Although there are more realistic models to accurately reflect the real behaviour of the material, in order to simplify the calculation involved, the idealized stress-strain curve is commonly adopted. In many applications, it is not necessary to use a sophisticated model, so a simplified constitutive law without work hardening is sufficient for the purposes here. This corresponds to the elastic-perfectly plastic material model, which can be seen in figure 2.

Figure 2: Model of elastic-perfect plastic material.



Source: Own authorship, 2024.

The load portion composed of the inclined lines plus the horizontal lines can be described by the following equations (Eq. (1) and Eq. (2)):

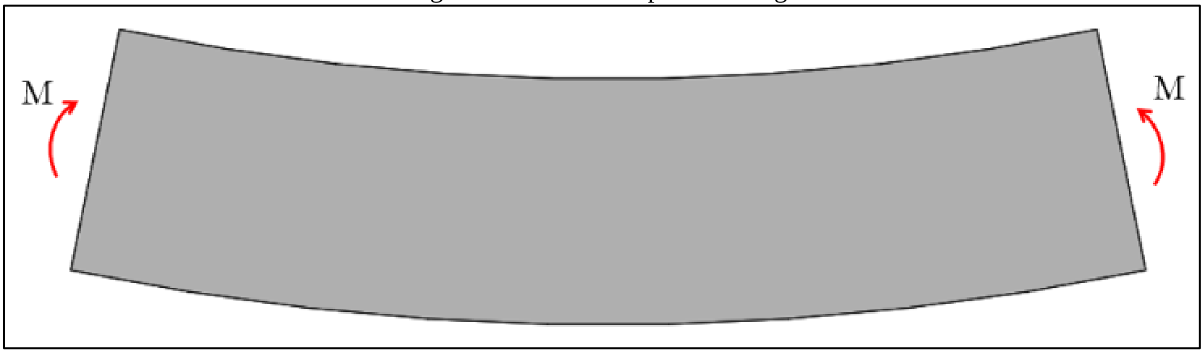
$$\sigma = E\, \varepsilon \quad , \quad |\varepsilon| < \frac{\sigma_y}{E}, \tag{1}$$

$$\sigma = \sigma_y\, sgn(\varepsilon) \quad , \quad |\varepsilon| \geq \frac{\sigma_y}{E}, \tag{2}$$

where $sgn(\varepsilon)$ takes the value if and if (compressive loading).$+1\varepsilon > 0 - 1\varepsilon < 0$

Therefore, plastic collapse exhibits the typical behavior shown in the last figure, which shows an indefinite increase in deflection under constant load. Once yielding is reached, an indefinite amount of stress can occur. This material model is only valid for sufficiently ductile materials. This applies to the cross-sections of structural elements subject to bending. For example, a beam (with two symmetrical axes in its cross-sectional plane) under bending moment applied at its ends is shown in Figure 3.M

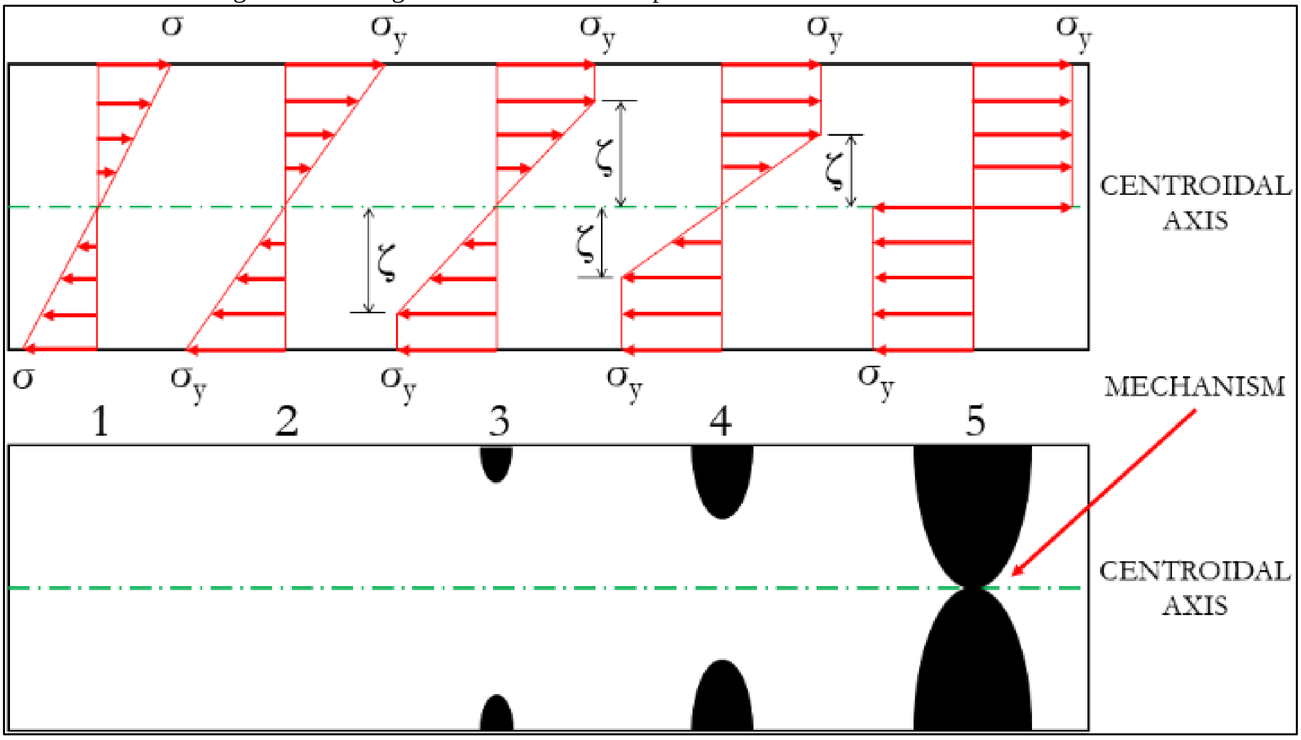
Figure 3: Beam under pure bending.



Source: Own authorship, 2024.

For an elastic-perfectly plastic material, the stress distribution over a cross-section sufficiently distant from the point of load application (observing the Saint-Venant principle) gradually assumes the stress distributions shown in the upper part of Figure 4. Although these distributions appear to represent the effect on different cross-sections across the longitudinal axis, this illustrates the same cross-section in subsequent stages of load increase (from left to right).

Figure 4: Bending distributions and their plastic behavior on a beam cross-section.



Source: Own authorship, 2024.

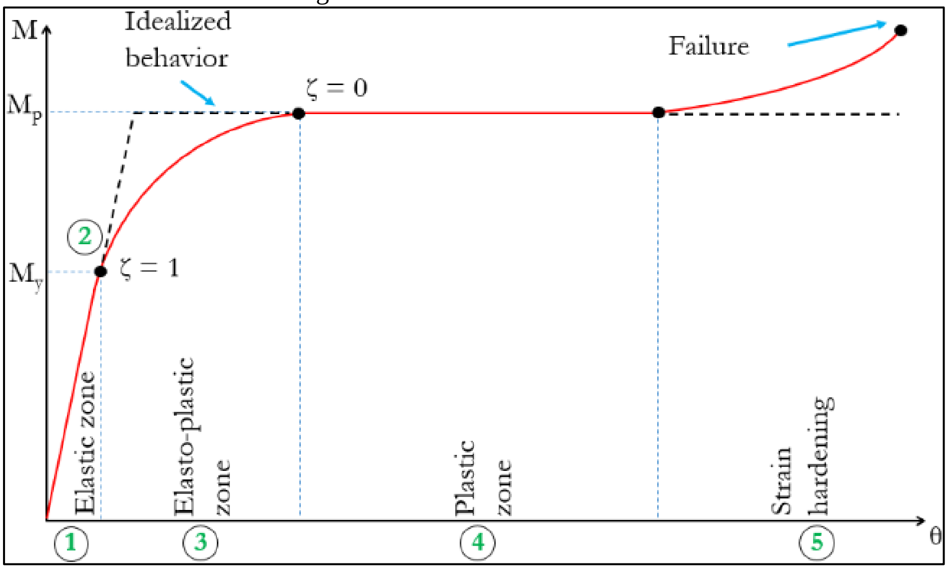
σ is the acting stress, such that; is the stress required to plasticize the beam material; is the height (from the centroidal axis) of the unplasticized surface.$\sigma < \sigma_y \sigma_y \xi$

In situation 1, the acting stress is still less than the elastic limit of the material () and therefore the cross-section analyzed only has elastic deformations. As the load increases, the stress also increases, so that at a certain point it reaches the yield strength of the material, causing plastic deformations (situation 2), since it is a perfectly plastic-elastic material. Initially, this phenomenon occurs in most of the outer fibers because they are under greater bending stresses (further away from the neutral axis). As this stress is never exceeded due to the assumption of this type of material, the adjacent fibers gradually reach the stress corresponding to the yield strength, spreading the plastic region (situations 3 and 4) and, consequently, decreasing the elastic region. If the loading continues to increase monotonically, the entire cross-section is plasticized (situation 5). At this point, the beam is said to experience plastic collapse. The plasticized cross-section acts as a plastic hinge (mechanism), providing freedom to twist around the axis of symmetry which is orthogonal to the plane of loading application.$\sigma < \sigma_y$

Schematically, the five stress situations are shown in the lower portion of Figure 4. In situations 1 and 2, as there is no plastic tension, the representation is simply not made. In situation 3, there is plastic deformation of most of the outer fibers. Therefore, plastic deformation begins to progress from the outermost fibers to the inner ones. This type of deformation affects the cross-section a little more in situation 4, reducing the elastic region; in 5, the entire cross-section is plasticized, thus creating a mechanism which, in this case, is called a plastic hinge.

The moment-rotation curve that represents these five stages is illustrated in figure 5. However, for the idealization of the elastic-perfectly plastic material, the graph is represented by the bilinear shape shown in the following figure as a dashed line.

Figure 5: Moment-rotation curve.



Source: Own authorship, 2024.

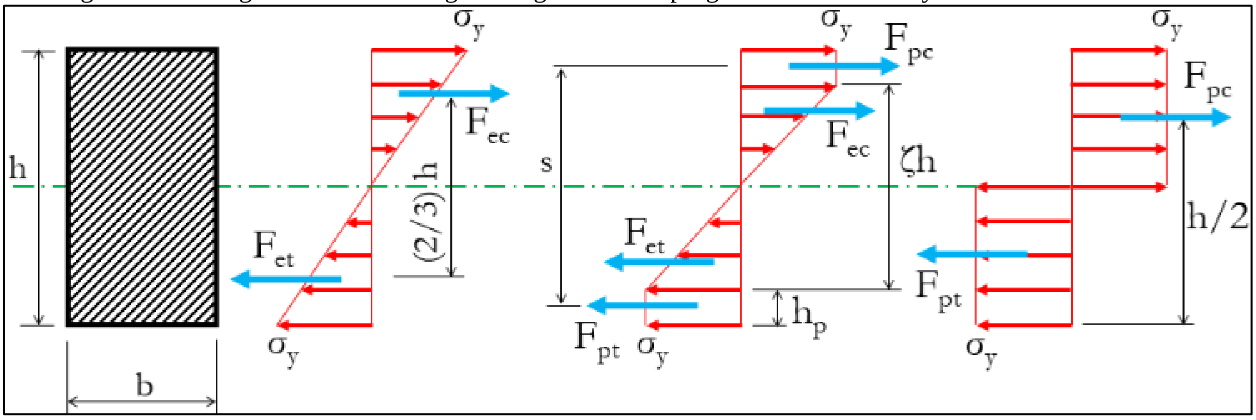
This idealization means that the cross-section linearly supports the bending moment up to the plastic moment capacity. Thereafter, the cross-section produces an undetermined amount of rotation.

The limit state (in which the body is still safe from collapse) is defined according to the situation immediately prior to collapse, i.e. it is a state imminent to plastic collapse. The load applied to reach the limit state is called the limit load.

As most structural systems are statically indeterminate, a local failure (local collapse) does not necessarily imply a global failure (total collapse). For these types of structures, a local collapse simply means that a structural constraint is no longer present, because a plastic hinge (for bending situations) has been created. Following this context, the load required to cause a local collapse is less than that required to cause global collapse. Therefore, structural collapse occurs when one or more local collapses occur, depending on the structural configuration.

Quantifying the additional load that a given cross-section can sustain beyond the yield point (when the fiber of the outermost material reaches yield strength) is relevant for the designer in order to select the most suitable cross-section geometry. So let and be the width and height of a rectangular cross-section, respectively. Recalling the stress distributions observed earlier, figure 6 indicates the forces and distances at each stage of a three-step transverse bending.bh

Figure 6: Bending distributions along the stages of stress progression in double-symmetrical cross-section.



Source: Own authorship, 2024.

From the stress distributions, the yield bending moment is given by Eq. (3),

$$M_y = F_{ec} \frac{2}{3} h \,, \tag{3}$$

where is the tensile force due to the plastic part of the tension; is the compressive force due to the plastic part of the tension; is the tensile force due to the elastic part of the tension; and is the compressive force due to the elastic part of the tension. When only the outermost fibers plasticize, this force is calculated from Eq. (4):$F_{pt}F_{pc}F_{et}F_{ec}$

$$F_{ec} = F_{et} = \frac{1}{2} \sigma_y \, b \, \frac{h}{2} \,. \tag{4}$$

Let the elastic section modulus be (exclusively a geometric property of the cross-section). Thus, in the case of a rectangular cross-section, the bending moment of the flow is expressed by Eq. (5):Z_e

$$M_y = \sigma_y \frac{b \, h^2}{6} = \sigma_y Z_e \,. \tag{5}$$

The elastoplastic bending moment is composed of the elastic () and plastic () parts, given by Eq. (6):$M^{ep} M_e^{ep} M_p^{ep}$

$$M^{ep} = M_e^{ep} + M_p^{ep} \,, \tag{6}$$

where the elastic component and the plastic part are respectively indicated by Eq. (7) and (8):

$$M_e^{ep} = \sigma_y \, \xi^2 \frac{b \, h^2}{6} \,, \tag{7}$$

$$M_p^{ep} = F_{pc} \, s \,, \tag{8}$$

where lever arm and are defined, respectively, according to Eq. (9) and Eq. (10):sh_p

$$s = \xi\, h + h_p \,, \tag{9}$$

$$h_p = \frac{h}{2}(1 - \xi)\,. \tag{10}$$

From then on, s is expressed as a function of ξ, as shown in Eq.(11):

$$s = \frac{h}{2}(1 + \xi)\,. \tag{11}$$

Therefore, in the case of a doubly symmetrical cross-section, the contribution of the plastic part of the force is given by Eq.(12):

$$F_{pc} = F_{pt} = \sigma_y\, b\, \frac{h}{2}\,(1 - \xi)\,. \tag{12}$$

Thus, Eq.(13) gives the plastic part of the bending moment:

$$M_p = \sigma_y\, \frac{b\, h^2}{4}\,(1 - \xi^2)\,. \tag{13}$$

Therefore, the elastoplastic bending moment results in the following relation (Eq.(14)):

$$M^{ep} = \sigma_y\, \frac{b\, h^2}{6}\, \frac{(3 - \xi^2)}{2}\,. \tag{14}$$

From the situation represented by the fully plasticized cross-section, the plastic bending moment and force are expressed by Eq.(15) and Eq.(16) respectively:

$$M_p = F_{pc}\, \frac{h}{2}\,, \tag{15}$$

$$F_{pc} = \sigma_y\, b\, \frac{h}{2}\,. \tag{16}$$

Let Z_p be the plastic section modulus of the cross-section. Thus, the plastic moment of the rectangular cross-section is expressed by Eq.(17):

$$M_p = \sigma_y\, \frac{b\, h^2}{4} = \sigma_y\, Z_p\,. \tag{17}$$

Therefore, the ratio between the plastic moment and the elastic moment, also called the shape factor f_s (dependent only on the geometry of the cross-section), is given by Eq.(18):

$$f_s = \frac{M_p}{M_y} = \frac{Z_p}{Z_e}. \tag{18}$$

Furthermore, in the specific case of rectangular cross-sections, the shape factor is indicated by Eq.(19):

$$f_s = \frac{\frac{b\,h^2}{4}}{\frac{b\,h^2}{6}} = \frac{3}{2}, \tag{19}$$

which means that this section can withstand fifty percent more bending than in the yield moment condition, before a plastic hinge occurs. Each type of cross-section has its own form factor. Thus, the form factor is a measure of the plastic's efficiency under bending.

METHODOLOGY

This section describes the strategy for applying the concepts already described to an example solved in two stages: (a) literally and conceptually; (b) numerically, in order to verify the increase in the capacity to withstand the bending moment in the context of an order. In addition, the example is solved by analytical calculation.

For this purpose, the example consists of a simply supported beam subjected to a concentrated load of medium span. Its cross-section is rectangular with base b, and height h, with the transverse load, P, applied at its centroidal point avoiding the phenomenon of torsion. The length of the span is defined by L.In the central region, the length of the elastoplastic region is defined by L_p.The aim is to obtain the elastic and plastic loads that can be applied without the system failing.

Tab.1 gives the numerical values for the parameters involved in the example.

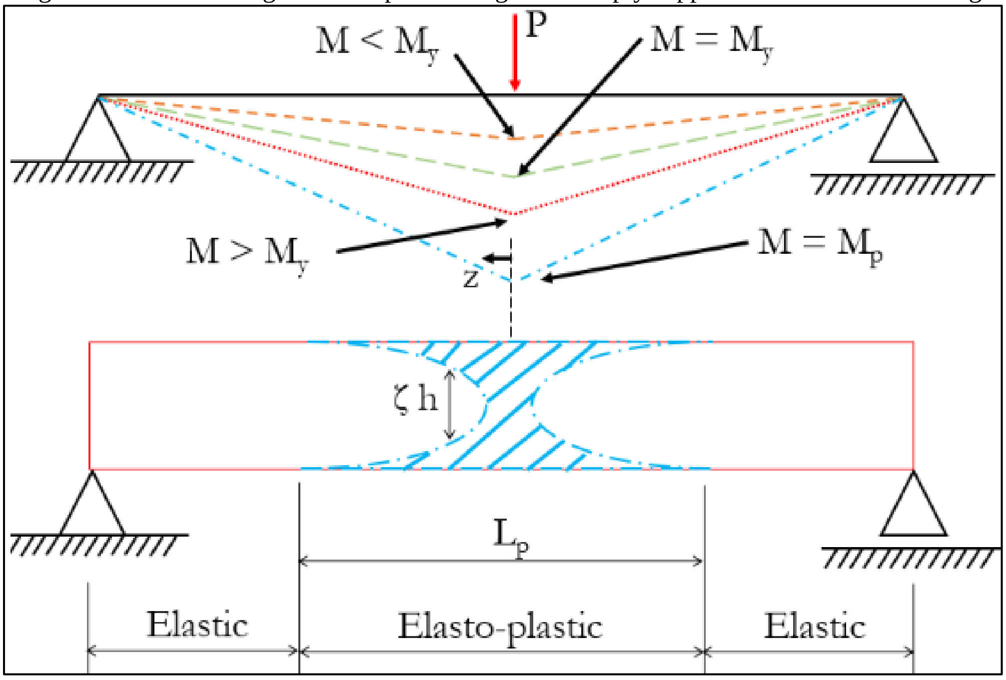
Table 1: Input data for the simply supported beam example.

Input data	**Variable**	**Value**
Rectangular cross-section base	b	$25\ mm$
Height of rectangular cross-section	h	$45\ mm$
Beam length	L	$1500\ mm$
Flow resistance	σ_y	$550\ MPa$

Source: author (2024).

Figure 7 shows this simply supported beam.

Figure 7: Plasticized region of the plastic hinge of a simply supported beam under bending.



Source: Own authorship, 2024.

RESULTS

The first step in the solution begins with the conceptual and literal approach. In the case of the simply supported beam, the bending moment in the middle of the beam span (at the location of the plastic hinge) is given by Eq. (20):

$$M_c = \frac{P\,L}{4}. \tag{20}$$

Thus, the load at which the yield occurs first is expressed by Eq. (21):

$$P_y = 4\frac{M_y}{L}. \tag{21}$$

The collapse load occurs when the critical bending moment at mid-span reaches the plastic moment capacity, which is represented by Eq. (22):

PP=4MPL. (22)

The relationship between collapse and yield load produces Eq. (23):

PpPy=4MpL4MyL=MpMy=fs . (23)

Therefore, for an isostatic structure, in which its structural members have the same cross-section (shape and dimensions), the ratio between the collapse load and the first yield load is simply the shape factor of the member's cross-section.f_s

The second part of the resolution refers to replacing the numerical input data to obtain the numerical output. From this point on, the calculation is based on the input data from Tab 1.

The elastic section modulus, Z_e in the case of a rectangular cross-section, is obtained from Eq. (24):

Ze=26=26=8437,5 3 . b h25 x 45mm (24)

Then, applying Eq. (5), the value of the bending moment of the flow is obtained from Eq. (25):

My=ye=550 x 8437.5=4,640,625 N mm .σZ (25)

The plastic section modulus is expressed by Eq. (26):

Zp=24=24=12656,25 3 . b h25 x 45mm (24)

Applying Eq. (17), the plastic bending moment is obtained from Eq. (27):

Mp=yp=550 x 8437.5=6,960,937.5 N mm .σZ (27)

According to Eq. (21), the maximum load that can be applied to reach the yield strength is calculated in Eq. (28):

Py=4MyL=44,640.6251500=12,375 N . (28)

According to Eq. (22), the maximum plastic load before failure is calculated in Eq. (29):

Pp=4MpL=46,960,937.51500=18,562.5 N . (29)

Just for checking purposes, Eq. (18) can be used to calculate the form factor (Eq. (30)),

fs=PpPy=18,562,512,375=1.5 , (30)

as expected.

CONCLUSIONS

This work takes up the concepts of elastic and plastic bending moment capacities of Timoshenko-Ehrenfest beams. The formulation is based simultaneously on the concepts of plasticity theory and a rectangular Timoshenko-Ehrenfest beam. The material model has been simplified to elastic-perfectly plastic. The stress distributions and the plastic hinges formed were associated to present the structural mechanism that causes rupture in a beam.

A simply supported beam was modeled in terms of the formulation presented and calculated for the input data provided. The form factor of the rectangular cross-section was confirmed through the calculations of the solved example.

Based on the concepts and theory presented in this work, a structural system can be analyzed in terms of an upper limit state, allowing the designer to increase the capacity of a structure composed of beam elements.

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